MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

Papers and Solutions not published in the "Educational Times."

EDITED BY

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MATHEMATICS

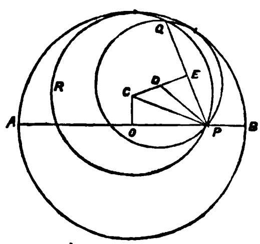
FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

1843. (Proposed by the EDITOR.)—Three points being taken at random within a circle, find the chance that the circle drawn through them will lie wholly within the given circle.

Solution by N'IMPORTE.



Let O be the centre; P and Q any two points within the given circle; and C and D the centres of the two circles drawn through P, Q, and tangential to the given one. Join OC, PC, PD, CD, producing CD (if necessary) to bisect PQ in E. Then if OB be the radius through P, the chance will obviously be the same whatever be the length of OB; therefore put OB = 1, OP $= x$, PQ $= y$, PC $= r$, $\angle$ OPQ $= \phi$; then

$$(1-r)^2 = OC^2 = x^2 + r^2 - 2rx\cos(\phi - CPE)$$
$$= x^2 + r^2 - xy\cos\phi - 2x\sin\phi\,(r^2 - \tfrac{1}{4}y^2)^{\frac{1}{2}},$$

whence we have this quadratic

$$4(1-x^2\sin^2\phi)\,r^2 - 4(1-x^2+xy\cos\phi)\,r + (1-x^2+xy\cos\phi)^2 + x^2y^2\sin^2\phi = 0,$$

the two roots of which must be the values of PC, PD; hence

$$PC + PD = \frac{1-x^2+xy\cos\phi}{1-x^2\sin^2\phi}.$$

Now Mr. Woolhouse has shown, in his Solution of this Question (*Reprint*, Vol. VIII., p. 91), that the required chance will be found by considering as the total number of cases those in which one point lies on the

circumference and the other two anywhere in the circle, that is, $= 3(2\pi)\pi^2 = 6\pi^3$; and as the favourable cases those in which the circle through the three points is tangential to the given circle; hence the third point R must lie on the circumference of one of the tangential circles, and therefore, while P and Q remain fixed, the number of positions R can take is expressed by

$$2\pi(\mathrm{PC}+\mathrm{PD}).$$

An element of the circle at P $= 2\pi x\,dx$, at Q $= y\,dy\,d\phi$; and the limits are x from 0 to 1, ϕ from 0 to π and doubled, y from 0 to $y' = (1-x^2\sin^2\phi)^{\frac{1}{2}} + x\cos\phi$; and the required chance is

$$p = \frac{1}{6\pi^3}\int_0^1 2\pi x\,dx \int_0^\pi 2d\phi \int_0^{y'} 2\pi y\,dy\,\frac{1-x^2+xy\cos\phi}{1-x^2\sin^2\phi}.$$

In integrating first with respect to y, from 0 to y', terms involving odd powers of $\cos\phi$ may be at once rejected, since these terms, having identical positive and negative values, must evidently disappear on afterwards integrating for ϕ; thus the integrations give

$$p = \frac{4}{3\pi}\int_0^1 x\,dx\int_0^\pi d\phi\left\{\tfrac{4}{3}x^2\cos^2\phi + \tfrac{2}{3}(1-x^2) - \frac{\tfrac{1}{3}(1-x^2)^2}{1-x^2\sin^2\phi}\right\}$$

$$= \frac{4}{3}\int_0^1 x\,dx\left\{\tfrac{2}{3} - \tfrac{1}{3}(1-x^2)^{\frac{3}{2}}\right\} = \tfrac{2}{5}.$$

[The result of this method of solution, compared with that before obtained by Messrs. Roberts and Woolhouse (*Reprint*, Vol. VIII., pp. 90—92), shows a proportion of $3:\pi$. That the process adopted in all these solutions is not quite rigid, will appear from the following explanatory Note by Mr. Woolhouse.]

NOTE ON QUESTION 1843. *By* W. S. B. WOOLHOUSE, F.R.A.S.

This Question, proposed by the Editor, is as follows:—

"Three points being taken at random within a circle, find the chance that the circle drawn through them will lie wholly within the given circle."

My Solution to this Question, contained in the *Reprint*, Vol. VIII. p. 91, concludes with this observation:—"This Solution is only approximative, the estimated points not being in strictness equally distributed. An accurate Solution of the Question would be very complicated." The object of the present Note is to explain more definitely why that Solution is only approximative, and what modification would in strictness be requisite to make it exact. It will bring us at once to the questionable part of the subject, and perhaps be more generally intelligible if we first reproduce so much of the Solution as may be considered unexceptionable, the same being contained in the following paragraph:—

The required chance, being obviously the same for all circles, will remain unaltered if the given circle be augmented by a concentric annulus; and

it will therefore follow that the same probability must result if the new cases thence arising be treated separately. It will be observed that the investigation is affected by this peculiarity, that the new cases which satisfy the proposed condition, are principally derived from a conversion of a portion of the old combinations which were previously unfavourable. Thus the newly acquired favourable cases are those in which the circle passing through any three points passes into the annulus without going beyond it; whereas the total new cases are those in which one, or more, of the three points is situated in the annulus. If we now conceive the annulus to be diminished without limit, the total new cases will ultimately become those which have one of the points in the periphery of the given circle, and the additional favourable cases will be those in which the circle drawn through the three points is tangential to the given circle.

The number of the additional favourable cases is thence estimated on the tacit assumption that the eligible points are equally distributed round the circumference of the tangential circle. Now this hypothesis of equal distribution, adopted on account of its expediency, is not strictly true. For let R be the radius of the given circle; $R + dR$ that of a consecutive concentric circle; and P_1, P_2 any two points on the surface; then if through these two points a circle radius ρ be drawn tangential to the given circle radius R, and another circle radius $\rho + d\rho$ tangential to the concentric circle radius $R + dR$, it will be evident that the third point P_3, on either side of the line P_1P_2 may be any point contained on the surface of the two differential lunes dL, dM contained between the circumferences of the circles radii ρ and $\rho + d\rho$, and the points of selection, not being included by an annulus of uniform width, are therefore not equally distributed round the circumference of either of those circles. When the point P_3 is situated in either differential lune dL or dM, it is obvious that the circumference of the circle $P_1P_2P_3$ must be wholly contained by those lunes, and therefore pass into the annulus without going beyond it.

Again, it is obvious that a second circle radius ρ' may be drawn through the points P_1, P_2 and touching the given circle on the other side of the line P_1P_2, and a like circle radius $\rho' + d\rho'$ touching the concentric circle. Between the circumferences of these we shall have two other differential lunes dL', dM', any point of which may equally serve as the third point P_3. But from the symmetry of the figure, those circles and lunes will, as P_1 and P_2 vary their positions, go through the same phases as the former, the values of which may therefore be doubled. Thus the total number of points P_3 producing the additional favourable cases is $2\Sigma \dfrac{dL + dM}{dR}$. And as the total new cases $= 3\dfrac{2\pi R dR}{dR} \times (\pi R^2)^2 = 6\pi^3R^5$, the required probability is to be found from the formula

$$p = \frac{1}{3\pi^3 R^5} \Sigma \frac{dL + dM}{dR} \text{..............................} (a),$$

where the summation under Σ is inclusive of all positions of P_1 and P_2.

The investigation may otherwise be accurately effected by taking the positions of P_3 intégrally. If through the two points P_1 and P_2 an indefinite number of circles be conceived to be drawn so that each may lie within the given circle, the extreme limiting positions of this system of circles will evidently be the two tangential circles radii ρ and ρ'. The integral lune L will be that portion of the area of the circle radius ρ which lies exterior to the circle radius ρ', and, *vice versâ*, the lune L′ will be that portion of the circle radius ρ' which lies exterior to the

circle radius a. If any point contained in either of the lunes L, L′ be taken as the third point P_3, the circumference of the circle $P_1P_2P_3$ will lie wholly within those lunes, and will therefore necessarily lie within the given circle. The integral number of favourable cases with respect to the point P_3 is hence correctly represented by the area L + L′, so that the probability, when P_1, P_2 are fixed, is $\frac{L+L'}{\pi R^2}$; and since, when P_1, P_2 vary, L and L′ pass through precisely the same phases of value, the required probability is

$$p = \frac{2\Sigma L}{(\pi R^2)^3} \quad \text{..............................} (\beta),$$

where, as before, the summation under Σ is inclusive of all positions of P_1 and P_2 on the surface of the given circle.

The formulæ (α) and (β) are both rigidly true, and, after the requisite integrations are accurately made, they must necessarily lead to identical results. By either method, the nature of the operations will, I presume, justify the opinion originally expressed, that an accurate solution of the Question would be very complicated. From the perfect symmetry and apparently elementary character of the problem, it is not improbable, however, that the final result may be comparatively simple, but at present I have no time to work it out, and must leave it in the able hands of other contributors. After all, however, it may be found only to confirm the accuracy of the result already arrived at by Mr. Roberts and myself, and there is some reason to favour such a conclusion, as the following mode of arriving at that result would appear to be free from the objection that has been pointed out:—

Let R denote the radius of the given circle, and x that of a circle passing through the three points. The number of ways, inclusive of permutations, in which the three points admit of being taken in the circumference $2\pi x$ is $(2\pi x)^3$. Also, when this circle radius x is wholly contained within the given one, its centre may evidently be anywhere on the area of a concentric circle radius $R-x$. Thus the number of favourable cases with respect to x is $\quad (2\pi x)^3 \times \pi (R-x)^2 = 8\pi^4 x^3 (R-x)^2$;

and for all values of x from 0 to R it is

$$8\pi^4 \int x^3 dx\, (R-x)^2 = \tfrac{2}{15}\pi^4 R^6.$$

The total cases being $(\pi R^2)^3$, the required probability is hence $\frac{2}{15}\pi$.

And when four points are taken within a sphere, an analogous process may be employed.

Note on Question 1843. *By* Samuel Roberts, M.A.

My Solution of this Question, given on p. 91 of Vol. VIII. of the *Reprint*, may be readily adapted to Mr. Woolhouse's ingenious suggestion, and the results are the same as his. Thus, adopting Mr. Woolhouse's view, let c be the distance of any point in the given circle from the centre, and let r be the radius of the given circle.

About the assumed point as centre, there can be drawn one interior tangential circle whose radius is $r-c$.

The number of ways in which three points can be taken on the circumference of this circle is $\frac{\{2\pi(r-c)\}^3}{2.3} = \frac{4\pi^3(r-c)^3}{3}$.

The element of the given circle at the assumed point is $2\pi c\,dc$. Hence the total number of favourable cases is given by

$$\frac{2\,.\,4}{3}\,\pi^4\int_0^r (r-c)^3\,c\,dc = \frac{2\pi^4 r^5}{15}.$$

But the total number of cases is given by

$$2\pi r\,.\,\tfrac{1}{2}(\pi r^2)^2 = \pi^3 r^5.$$

Therefore the required probability is $\frac{2}{15}\pi$.

Exactly in the same way, we get $\frac{3}{35}\pi$ for the sphere, and the expression already given for the hyper-sphere.

It is worth remarking that the general solution is

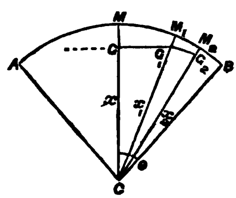
$$\frac{n^n\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n+1)}\,\frac{1\,.\,2\,\ldots\,(n^2-1)\,n^2}{n\,(n+1)\,\ldots\,(n+n^2)}.$$

2975. (Proposed by Professor SYLVESTER.)—Prove that a body acted on by forces tending to two fixed centres, and varying as the inverse fifth power of the distances, may be made to describe a circle.

Solution by the REV. J. WOLSTENHOLME, M.A.

If a particle can describe a given orbit under a force F, and the same orbit under a force F′, it can also describe the same orbit under the simultaneous action of the forces F and F′, and so for any number of forces; the *vis viva* of the particle at any point of the orbit under the action of the combined forces being the sum of those at the same point under the forces separately. Hence, if a body be acted on by two forces to fixed centres, each varying inversely as the fifth power of the distance, it can be set in motion at any point in such a way as to describe a circle through the two centres of force.

ELEMENTARY METHOD OF FINDING THE CENTRE OF GRAVITY OF A CIRCULAR ARC. *By* M. W. CROFTON, F.R.S.



Let AB be the arc, 2θ the angle it subtends at the centre, G its centre of gravity. Bisect the arc MB in M_1. Now, if G_1 be the centre of gravity of MB, it is easy to see that G_1G is perpendicular to CG. Bisect M_1B again in M_2, and so on; let the distances of the series of centres of gravity G, G_1, G_2, ..., from the centre, be x, x_1, x_2, ...

$$x_1 = \frac{x}{\cos\frac{1}{2}\theta},\quad x_2 = \frac{x}{\cos\frac{1}{2}\theta\,\cos\frac{1}{4}\theta},\quad \text{\&c.}$$

But, ultimately, $x_n = r$; therefore $r = \dfrac{x}{\cos\frac{1}{2}\theta\cos\frac{1}{4}\theta\cos\frac{1}{8}\theta\ldots}$.

Now $\cos\frac{1}{2}\theta\cos\frac{1}{4}\theta\cos\frac{1}{8}\theta\ldots = \dfrac{\sin\theta}{\theta}$; therefore $x = r\dfrac{\sin\theta}{\theta}$.

2954. (Proposed by the Rev. J. Wolstenholme, M.A.)—The sides of a triangle ABC are a, b, c, and of A′B′C′ are $b+c$, $c+a$, $a+b$, also the angles B, B′ are equal; prove that

$$\cos\tfrac{1}{2}(\mathrm{C}-\mathrm{A}) = 4\sin\tfrac{1}{2}\mathrm{B} - \sin\tfrac{3}{2}\mathrm{B}.$$

Solution by R. Tucker, M.A.

From the triangle A′B′C′ we have $s'=2s$, $s'-a'=a$, $s'-b'=b$, $s'-c'=c$;

therefore $\qquad \tan^2\frac{1}{2}\mathrm{B}' = \dfrac{ac}{bs'} = \dfrac{\sin\mathrm{A}\sin\mathrm{C}}{\sin\mathrm{B}\,(\sin\mathrm{A}+\sin\mathrm{B}+\sin\mathrm{C})}$,

or $\qquad 2\sin^2\frac{1}{2}\mathrm{B}' = \dfrac{\sin\mathrm{A}\sin\mathrm{C}\cos\frac{1}{2}\mathrm{B}}{4\cos\frac{1}{2}\mathrm{A}\cos\frac{1}{2}\mathrm{B}\cos\frac{1}{2}\mathrm{C}} = \sin\frac{1}{2}\mathrm{A}\sin\frac{1}{2}\mathrm{C}$.

Now $\quad 4\sin\frac{1}{2}\mathrm{B} - \sin\frac{3}{2}\mathrm{B} = \sin\frac{1}{2}\mathrm{B} + 4\sin^3\frac{1}{2}\mathrm{B}$

$$= \cos\tfrac{1}{2}(\mathrm{A}+\mathrm{C}) + 2\sin\tfrac{1}{2}\mathrm{A}\sin\tfrac{1}{2}\mathrm{C} = \cos\tfrac{1}{2}(\mathrm{C}-\mathrm{A}).$$

3017. (Proposed by A. W. Panton, B.A.)—Prove that, for the law of the inverse square, the attraction of a homogeneous ellipsoid, determined by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0,$$

on any point situated on the cone

$$(2a^2-b^2-c^2)\,x^2(2b^2-c^2-a^2)\,y^2(2c^2-a^2-b^2)\,z^2 = 0,$$

is the same as if the whole mass of the ellipsoid were condensed into its centre.

Solution by G. M. Minchin, B.A.

Mr. Panton's theorem is not quite accurate in the form in which it is stated. It has reference to those points only which are at a considerable distance from the ellipsoid.

That it is true for such points appears thus :—

Laplace assumes the potential V to be expanded in a series in $(\mathrm{U}_0+\mathrm{U}_1+\mathrm{U}_2+\ldots+\mathrm{U}_i)$, ascending in the constants of the ellipsoid, and descending in the coordinates of the attracted point. This is of itself sufficient to show that the point must be at a considerable distance from the ellipsoid.

Laplace then deduces the following equation, by means of which each term of the above series may be calculated from the preceding term:

$$(i+1)(2i+5)\,r\,U_{i+1} = (2i+1)\,\theta(\varpi-\theta)\frac{dU_i}{d\theta} - (2i+\tfrac{3}{2})\,\theta.y\frac{dU_i}{dy}$$

$$-(2i+\tfrac{3}{2})\,\varpi z\frac{dU_i}{dz} - \tfrac{1}{2}(2i+1)\left\{\theta+(2i+1)\varpi\right\}U_i,$$

in which r = distance of attracted point from centre of ellipsoid,
$\theta = b^2-a^2$, and $\varpi = c^2-a^2$.

Now $U_0 = \dfrac{1}{r}$, therefore $5rU_1 = \dfrac{3}{2}\cdot\dfrac{\theta y^2}{r^3}+\dfrac{3}{2}\cdot\dfrac{\varpi z^2}{r^3}-\dfrac{\theta+\varpi}{2r}$.

It is evident now, from the equation connecting U_i and U_{i+1}, that if $U_1=0$ identically, we shall have $U_2=U_3=$ &c. $=0$; and then we shall have simply

$$V = mU_0 = \frac{m}{r}.$$

Putting $U_1=0$, and substituting for r, θ, ϖ their values, we have

$$(2a^2-b^2-c^2)\,x^2+(2b^2-c^2-a^2)\,y^2+(2c^2-a^2-b^2)\,z^2 = 0.$$

That the theorem does not hold for points on the surface, is evident from the fact that, for a point on the surface, the potential can be expressed in the form $m(Lx^2+My^2+Nz^2)$, L, M, N being certain constant definite integrals. It is easy to show that this form could never agree with $\dfrac{m}{(x^2+y^2+z^2)^{\frac{1}{2}}}$. Moreover, the fact that the attraction of a "couche" on an external point is normal to the confocal couche passing through that point, is sufficient to show that for elliptical couches the attracted points must be very distant. And as the theorem does not hold for points on the surface, it cannot hold for internal points.

2967. (Proposed by the Rev. J. Wolstenholme, M.A.)—Integrate the equations in differences

(1) .... $2(u_{x+1}-u_x)^3 = (u_{x+1}+2u_x)(u_x+2u_{x+1})$,

(2) .... $(u_{x+1}-u_x)^2 = u_{x+1}+u_x$.

Solution by the Proposer.

1. Take $$v_x \equiv \frac{3u_x}{u_{x+1}-u_x};$$

therefore $$v_x+1 = \frac{u_{x+1}+2u_x}{u_{x+1}-u_x}, \quad\text{and}\quad v_x+2 = \frac{2u_{x+1}+u_x}{u_{x+1}-u_x};$$

therefore $$v_x(1+v_x)(2+v_x) = 6u_x,$$

and $\quad u_{x+1}-u_x = \dfrac{(1+v_x)(2+v_x)}{2}$

or $\quad u_{x+1}-u_x = \frac{1}{6}\left\{v_{x+1}(1+v_{x+1})(2+v_{x+1})-v_x(1+v_x)(2+v_x)\right\}$,

or $\quad v_{x+1}(1+v_{x+1})(2+v_{x+1}) = (1+v_x)(2+v\ )(3+v\ )$,

which is satisfied by $\quad v_{x+1} = 1+v_x$, or $v\ = x+\mathrm{C}$,

and therefore $\quad u_x = \frac{1}{6}(x+\mathrm{C})(x+\mathrm{C}+1)(x+\mathrm{C}+2)$.

This is a perfectly general solution, involving the finite difference constant C in the third degree; but it is not the only solution, as we may have

$$v_{x+1}^2 + v_{x+1}(4+v_x)+6+5v_x+v_x^2 = 0.$$

2. If we assume

$$v_x = \frac{2u_x}{u_{x+1}-u_x},\quad 1+v_x = \frac{u_{x+1}+u_x}{u_{x+1}-u_x},\quad \text{therefore}\quad v_x(1+v_x) = 2u_x;$$

and $\quad 1+v_x = u_{x+1}-u_x = \frac{1}{2}\left\{v_{x+1}(1+v_{x+1})-v_x(1+v_x)\right\}$,

or $v_{x+1}(1+v_{x+1}) = (1+v_x)(2+v_x)$, therefore $v_{x+1} = 1+v_x$ or $-(2+v_x)$;

of which the first gives $\quad u\ = \frac{1}{2}(x+\mathrm{C})(x+\mathrm{C}+1)$,

and the second, $\quad 2u\ = \mathrm{C}(-1)^x\left\{-1+\mathrm{C}(-1)^x\right\}$,

and each of these is a perfectly general solution of the given equation.

2966. (Proposed by Matthew Collins, B.A.)—If four circles touch each other, and if three of them touch a straight line; prove that the distance of this straight line from the centre of the fourth circle is equal to seven times its radius.

I. *Solution by the* Rev. J. Wolstenholme, M.A.

If three circles touch each other and a straight line touch the three, their contact must be external: let A, B, C be the centres, a, b, c the radii; O the centre, and r the radius of a circle touching the three; and α, β, γ the angles BOC, COA, AOB. Then

$$\cos\alpha = 1-\frac{2bc}{(b+r)(c+r)},\ \&\text{c.},\quad\text{and}\quad \cos^2\alpha+\ \dots\ -2\cos\alpha\cos\beta\cos\gamma = 1.$$

The equation for r is then

$$b^2c^2(a+r)^2+\ \dots\ -2a^2bc(b+r)(c+r)-\ \dots\ +4a^2b^2c^2 = 0,$$

or $\quad r^2\left\{b^2c^2+c^2a^2+a^2b^2-2abc(a+b+c)\right\}-2rabc(bc+ca+ab)+a^2b^2c^2 = 0.$

Since a straight line touches the three, one of these values of r must be infinite, whence

$$\left(\frac{1}{a}\right)^{\frac12}+\left(\frac{1}{b}\right)^{\frac12}+\left(\frac{1}{c}\right)^{\frac12}=0, \quad \text{and} \quad r=\frac{abc}{2\,(bc+ca+ab)} \quad \text{for the finite circle.}$$

If (X, Y, Z) be areal coordinates of O measured on the triangle ABC, the perpendicular p from O on the straight line is $a\mathrm{X}+b\mathrm{Y}+c\mathrm{Z}$; and

$$\begin{aligned}\mathrm{X}:\mathrm{Y}:\mathrm{Z} &= (b+r)\,(c+r)\sin\alpha : (c+r)\,(a+r)\sin\beta : (a+r)\,(b+r)\sin\gamma \\ &= \left(\frac{b+c+r}{a}\right)^{\frac12} : \left(\frac{c+a+r}{b}\right)^{\frac12} : \left(\frac{a+b+r}{c}\right)^{\frac12} \\ &= b+c\pm\tfrac12(bc)^{\frac12} : c+a\pm\tfrac12(ca)^{\frac12} : a+b\pm\tfrac12(ab)^{\frac12},\end{aligned}$$

reducing by the relation between a, b, c; the sign of the ambiguity $\pm(bc)$ being that of the product of $\left(\frac{1}{b}\right)^{\frac12}$ and $\left(\frac{1}{c}\right)^{\frac12}$ when the relation is satisfied.

Hence $$p=\frac{2\,(bc+ca+ab)+\tfrac12\left\{\pm a\,(bc)^{\frac12}\pm b\,(ca)^{\frac12}\pm c\,(ab)^{\frac12}\right\}}{2\,(a+b+c)};$$

but since $(bc)^{\frac12}+(ca)^{\frac12}+(ab)^{\frac12}=0, \quad bc+ca+ab+2\left\{\pm a\,(bc)^{\frac12}\pm\dots\right\}=0,$

whence $$p=\frac{2\,(bc+ca+ab)-\tfrac14(bc+ca+ab)}{2\,(a+b+c)}=\frac78\,\frac{bc+ca+ab}{a+b+c}$$

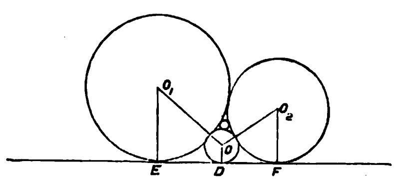
$$=\frac72\,\frac{abc}{bc+ca+ab} \text{ (by relation between } a, b, c) = 7r.$$

It will be found in general that the perpendiculars from A on the common tangents to the other two circles are

$$\frac{2bc\,(b+c)-a\,(b-c)^2\pm 4bc\left\{a\,(a+b+c)\right\}^{\frac12}}{(b+c)^2},$$

one value of which is a when the condition $(bc)^{\frac12}+(ca)^{\frac12}+(ab)^{\frac12}=0$ is satisfied.

II. *Solution by* STEPHEN WATSON; R. TUCKER, M.A.; *and others.*



Let O, O_1, O_2 be the centres; r, r_1, r_2 the radii of the three circles; and D, E, F the points of contact with the line. Then

$$\mathrm{DE}=\left\{(r_1+r)^2-(r_1-r)^2\right\}^{\frac12}$$
$$=2(r_1r)^{\frac12},$$

$\mathrm{DF}=2\,(r_2r)^{\frac12}$, $\mathrm{EF}=2(r_1r_2)^{\frac12}$

therefore $$(r_1r)^{\frac12}+(r_2r)^{\frac12}=(r_1r_2)^{\frac12} \text{ ....(1).}$$

Take DF, DO for axes, denote the centre of the fourth circle by (x, y) and its radius by ρ; then

$$(x+\mathrm{ED})^2+(y-r_1)^2=(r_1+\rho)^2 \text{ ....(2),}$$
$$x^2+(y-r)^2=(r+\rho)^2 \text{ ....(3),}$$
$$(x-\mathrm{DF})^2+(y-r_2)^2=(r_2+\rho)^2 \text{ ....(4)}$$

Take (3) first from (2) and then from (4); the results give

$$2\mathrm{DE}\,.\,x+\mathrm{DE}^2 = 2\,(y+\rho)\,(r_1-r) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

$$-2\mathrm{DF}\,.\,x+\mathrm{DF} = 2\,(y+\rho)\,(r_2-r) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Eliminating x, and putting for DE, DF their values above, the result divides by $(r_1r)^{\frac{1}{2}}+(r_2r)^{\frac{1}{2}}$, and we have

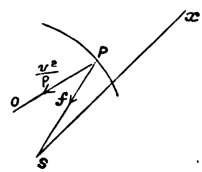
$$y+\rho = \frac{2r\,(r_1r_2)^{\frac{1}{2}}}{(r_1r_2)^{\frac{1}{2}}-r} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

Hence, by (5), $x = \dfrac{r^{\frac{3}{2}}\,(r_1^{\frac{1}{2}}-r_2^{\frac{1}{2}})}{(r_1r_2)^{\frac{1}{2}}-r}$, and (3) gives

$$y-\rho = 2r-\frac{x^2}{y+\rho} = \text{[when reduced by (1)]}\;\frac{3r\,(r_1r_2)^{\frac{1}{2}}}{2\left\{(r_1r_2)^{\frac{1}{2}}-r\right\}};$$

therefore $\quad y+\rho = \frac{4}{3}\,(y-\rho),\quad$ therefore $\quad y = 7\rho.$

Investigation of the Equation of Motion of a Particle under a Central Force. *By* C. R. Rippin, M.A.



Take O the centre of curvature, then the acceleration in PO $= \dfrac{v^2}{\rho}$. Let f be the acceleration to the centre S; then $\dfrac{v^2}{\rho} = f\sin\phi = fpu$,

$$\rho = \frac{\left\{u^2+\left(\frac{du}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{u^3\left(u+\frac{d^2u}{d\theta^2}\right)} = \frac{1}{p^3u^3\left(u+\frac{d^2u}{d\theta^2}\right)};$$

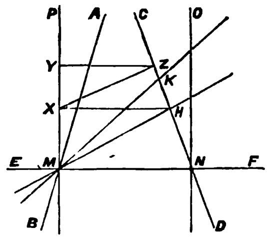
and $\quad v^2 = \dfrac{h^2}{p^2};\quad$ therefore $\quad u+\dfrac{d^2u}{d\theta^2} = \dfrac{f}{h^2u^2},$ the equation required.

3034. (Proposed by J. J. Walker, M.A.)—Let two conics S, S′ be inscribed in the same quadrilateral; then the anharmonic ratio of the four points, in which any tangent to the *former* conic is cut by the four sides of the quadrilateral, is equal to that of the pencil formed by joining the four points of intersection of S and S′ with any fifth point on the *latter* conic.

Solution by the Rev. R. Townsend, F.R.S.; J. P. Taylor, M.A.; *and others.*

For, either anharmonic ratio reciprocating into the other to each of the four conics with respect to which S and S′ are reciprocal polars to each other, therefore, &c.

Note on Euclid's 12th Axiom. *By* W. Hanna.



Let AB, CD be two straight lines, and let EF, falling on them, make the two interior angles AMN, CNM together less than two right-angles; then AB, CD will meet towards A, C, if continually produced.

The line EF may make with the two lines three different sets of angles:—

(1) when each of the interior angles AMN, CNM is less than a right-angle;

(2) when one, as CHM, is a right-angle, and the other, AMH, acute;

(3) when one, CKM, is obtuse, and the other, AMK, acute.

At the points M, N draw MP, NO perpendicular to EF; then, since the angle CNM is acute, the perpendicular MH will meet CN on the side towards C. Draw the perpendiculars HX, XZ, ZY, &c. to PM, CN.

Now, since the angle HNM is less than MHN, MN is greater than MH (Euc. I. 19); and because HMX is less than the right-angle HXM, MH is greater than HX. Thus the perpendiculars XZ, ZY, &c., drawn between PM and CD, decrease in length, so that the perpendicular will ultimately become less than any assignable line. Hence the line CN will approach PM to a distance less than anything conceivable. In the same way it may be shown that AM approaches ON in a like manner. But AB, CD must cut each other before either can fulfil the foregoing conditions; therefore AB will meet CD.

In like manner, it may be proved in the second and third cases that AB, CD will cut each other.

3025. (Proposed by J. J. Walker, M.A.)—Three lines in the same plane make, with any axis which they meet, angles $\tan^{-1} m_1$, $\tan^{-1} m_2$, and ϕ respectively, m_1 and m_2 being the roots of $am^2+\beta m+\gamma=0$; prove that the product of the sines of the angles which the first two lines make with the third is given by the formula

$$\frac{a\sin^2\phi+\beta\sin\phi\cos\phi+\gamma\cos^2\phi}{\left\{(a-\gamma)^2+\beta^2\right\}^{\frac{1}{2}}}.$$

Solution by A. A. Bourne; R. W. Genese; *and others.*

Let OA, OB, OC be three lines making with OX angles $\tan^{-1} m_1$, $\tan^{-1} m_2$, ϕ. Then, since m_1, m_2 are roots of the equation $am^2+\beta m+\gamma=0$, we have

$$(m_1+m_2)=-\frac{\beta}{a},\qquad m_1m_2=\frac{\gamma}{a}.$$

Now $$\sin \text{AOC} = \sin\phi\cos \text{AOX} - \cos\phi \sin \text{AOX}$$

$$=(\sin\phi - m_1\cos\phi)\,(1+m_1^2)^{-\frac{1}{2}},$$

also $\sin BOC = (\sin\phi - m_2\cos\phi)(1+m_2^2)^{-\frac{1}{2}}$;

therefore $\sin AOC \, . \, \sin BOC = \dfrac{\sin^2\phi + m_1 m_2 \cos^2\phi - (m_1+m_2)\sin\phi\cos\phi}{(1+m_1^2+m_2^2+m_1^2m_2^2)^{\frac{1}{2}}}$

$$= \frac{\sin^2\phi + \dfrac{\beta}{\alpha}\sin\phi\cos\phi + \dfrac{\gamma}{\alpha}\cos^2\phi}{\left(1+\dfrac{\beta^2}{\alpha^2} - 2\dfrac{\gamma}{\alpha} + \dfrac{\gamma^2}{\alpha^2}\right)^{\frac{1}{2}}} = \frac{\alpha\sin^2\phi + \beta\sin\phi\cos\phi + \gamma\cos^2\phi}{\left\{(\alpha-\gamma)^2+\beta^2\right\}^{\frac{1}{2}}}.$$

[By means of this formula and those given by Mr. WOLSTENHOLME in his Solution of Case 1 of Quest. 2942, a similar proof of Case 2 has been given by Mr. WALKER in the *Reprint*, Vol. XII., p. 100.]

3020. (Proposed by the Rev. J. BLISSARD.)—The formula given by Professor Sylvester in Question 2977 may be put under the form

$$\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\dots+\frac{1}{2x-1} = \frac{x}{2x-1} + \frac{2}{2}\cdot\frac{x(x-1)}{(2x-1)(2x-3)} + \frac{2^2}{3}\cdot\frac{\&c.}{\&c.}+\dots.$$

The following generalisation is proposed for solution:—

$$\frac{1}{1}+\frac{1}{m+1}+\frac{1}{2m+1}+\dots+\frac{1}{mx+m+1}$$

$$= \frac{x}{mx-m+1} + \frac{m}{2}\cdot\frac{x(x-1)}{(mx-m+1)(mx-2m+1)} + \frac{m^2}{3}\cdot\frac{\&c.}{\&c.}+\dots.$$

Solution by J. J. WALKER, M.A.

My Solution of Question 2977 (*Reprint*, Vol. XII., p. 84) may be readily adapted to prove the above generalisation, as follows:—

$$\frac{d^x y^{-\frac{1}{m}}\log_e y}{dy^x} = \frac{(-1)^{x-1}}{m^{x-1}} \, . \, 1 \, . \, (m+1)(2m+1)\dots\dots$$

$$\dots\dots(mx-2m+1)(mx-m+1)\left(\frac{1}{1}+\frac{1}{m+1}+\frac{1}{2m+1}+\dots\dots\right.$$

$$\left.\dots+\frac{1}{mx-m+1}-\frac{\log_e y}{m}\right)\dots\dots(1);$$

and this divided by $1.2.3\dots x$, y being put equal to 1, is the coefficient of z^x in the development of $(1+z)^{-\frac{1}{m}}\log_e(1+z)$, which, by actual multiplication of the expansions of $(1+z)^{-\frac{1}{m}}$ and $\log_e(1+z)$, is

$$\left(\frac{-1}{m}\right)^{x-1}\frac{(m+1)(2m+1)\dots(mx-m+1)}{1.2.3\dots x}\left(\frac{x}{mx-m+1}\right.$$

$$\left.+\frac{m}{2}\frac{x(x-1)}{(mx-m+1)(mx-2m+1)}+\frac{m^2}{3}\cdot\frac{\&c.}{\&c.}\dots\dots\frac{\&c.}{\&c.}\text{ to } x \text{ terms}\right)\dots\dots\dots(2).$$

Equating (1) divided by $1.2.3\dots x$, after putting $y=1$, with (2), the generalisation proposed is proved.

2822. (Proposed by A. MARTIN.) — A raffling match is composed of 5 persons, each throwing 3 times with 7 pennies, the one turning up the greatest number of heads to be winner. The third player having turned up 15 heads, it is required to determine his chance of winning.

Solution by the PROPOSER.

Throwing 3 times with 7 pennies is the same as throwing once with 21 pennies, or 21 times with 1 penny.

Let $m=21$, $n=$ the number of players, and $a=$ the number of heads turned up by the rth player.

The chance of turning any penny head up is $\frac{1}{2}$, and the chance of turning up a heads in one throw with m pennies is

$$\frac{m(m-1)(m-2)\dots(m-a+1)}{1.2.3\dots a.2^m}=\frac{m(m-1)(m-2)\dots(a+1)}{1.2.3\dots(m-a).2^m}=\frac{6783}{262144},$$

when $m=21$ and $a=15$, which put $=p$.

The chance of turning up fewer than a heads is the sum of the chances of turning up 0, 1, 2, 3, ... $(a-1)$ heads; that is,

$$\frac{1}{2^m}+\frac{m}{1.2^m}+\frac{m(m-1)}{1.2.2^m}+\dots\dots+\frac{m(m-1)(m-2)\dots(m-a+2)}{1.2.3\dots(a-1).2^m}=\frac{251874}{262144},$$

when $m=21$ and $a=15$, which put $=q$.

$$p+q=\frac{258657}{262144},\text{ the chance of turning up not more than 15 heads.}$$

Before the rth player tried his chance, it was the same as the chance of any of the other $(n-r)$ players, who, with himself, made $(n-r+1)$, which put $=s$.

The chance that none of these s players will turn up more than a heads is $(p+q)^s$; the chance that some of them will turn up a heads is $(p+q)^s-q^s$; and the chance that any particular individual will turn up a heads is $\{(p+q)^s-q^s\}\div s$.

But the rth player *has* turned up a heads, the chance of which was p; therefore his chance of winning on that number after it is played is

$$\frac{(p+q)^s-q^s}{sp}=\frac{(p+q)^3-q^3}{3(p+q-q)}=\frac{(p+q)^2+(p+q)q+q^2}{3}=\frac{65164309581}{68719476736}.$$

2837. (Proposed by MATTHEW COLLINS, B.A.)—If the numbers a, b, c be the sides of a triangle, prove that

$$\tfrac{2}{3}(a+b+c)(a^2+b^2+c^2) > a^3+b^3+c^3+3abc.$$

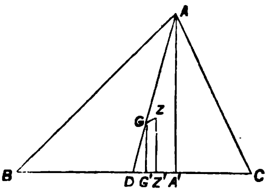
I. *Solution by* ASHER B. EVANS, M.A.

We may obviously suppose $a>b>c$, and put $a=c+m+n$, $b=c+m$; where $c>n$, and m and n are positive. Then, by substitution, we find

$$\tfrac{2}{3}(a+b+c)(a^2+b^2+c^2)-(a^3+b^3+c^3+3abc)$$
$$=\tfrac{1}{3}(c-n)(mn+n^2)+\tfrac{1}{3}cm^2+\tfrac{2}{3}m^3+m^2n.$$

Since $c>n$ and m and n are positive, the right-hand member of this equation is positive; therefore $\tfrac{2}{3}(a+b+c)(a^2+b^2+c^2) > a^3+b^3+c^3+3abc.$

II. *Solution by the* PROPOSER.



1. Let ABC be a triangle, AA′ its altitude, D the middle point of BC, DG $=\tfrac{1}{3}$DA, and therefore the perpendicular GG′ $=\tfrac{1}{3}$AA′, G being the centre of gravity. Let Z be the centre and ZZ′ the radius of its inscribed circle; then we have

$$ZZ'-GG'=\frac{\Delta}{3as}(3a-2s)$$
$$=\frac{\Delta}{3as}(2a-b-c),$$

and the projection of GZ on BC is

$$G'Z'=DZ'-\tfrac{1}{3}DA'=\frac{c-b}{2}-\frac{c^2-b^2}{6a}=\frac{c-b}{6a}(3a-b-c);$$

$$\therefore\ GZ^2=G'Z'^2+(ZZ'-GG')^2=\frac{\Delta^2}{9a^2s^2}(2a-bc-c)^2+\frac{(c-b)^2}{36a^2}(3a-b-c)^2=$$

$$\frac{1}{72a^2s}\Big\{(b+c-a)(a+c-b)(a+b-c)(2a-b-c)^2+(a+b+c)(b-c)^2(3a-b-c)^2\Big\}$$
$$=\frac{1}{18s}(-a^3-b^3-c^3+2b^2c+2bc^2+2a^2b+2ab^2+2a^2c+2ac^2-9abc)$$
$$=\frac{1}{18s}\Big\{(a+b+c)(3ab+3bc+3ca-a^2-b^2-c^2)-18abc\Big\}$$
$$=\frac{1}{9}(3ab+3bc-3ca-a^2-b^2-c^2)-\frac{2abc}{a+b+c}.$$

2. If a, b, c be the sides of a triangle, we see from the above that

$$2\Sigma(a^2b) > a^3+b^3+c^3+9abc;$$

adding $2a^3+2b^3+2c^3$ to each, we have

$$2(a+b+c)(a^2+b^2+c^2) > 3a^3+3b^3+3c^3+9abc,$$

therefore $\tfrac{2}{3}(a+b+c)(a^2+b^2+c^2) > a^3+b^3+c^3+3abc.$

The theorem is sometimes true when a, b, c are not the sides of a triangle; for instance, it is true when $a=2$, $b=3$, $c=6$, but not when $a=2$, $b=3$, $c=7$.

2992. (Proposed by R. TUCKER, M.A.)—If A be the point on an ellipse through which the osculating circles at B, C, D pass, and m, m' the tangents of the angles which the tangents at A, and B, C, D make with the major axis, then $$m'^3+3m'^2m-3m'(1-e^2)-m(1-e^2)=0.$$

Solution by REV. T. J. SANDERSON, M.A.; R. W. GENESE; *the* PROPOSER; *and others.*

If ϕ be the eccentric angle of any of the points B, C, D, and θ that of A, we have the relation $\tan 3\phi = -\tan\theta$.

But $$\tan\theta = -\frac{b}{am}, \text{ and } \tan\phi = -\frac{b}{am'};$$

therefore $$\frac{\left(\frac{b}{am'}\right)^3 - 3\frac{b}{am'}}{1-3\left(\frac{b}{am'}\right)^2} = \frac{b}{am},$$

or $$m'^3+3m'^2m-3m'(1-e^2)-m(1-e^2)=0.$$

2911. (Proposed by F. D. THOMSON, M.A.)—P is a fixed point on the outer of two confocal ellipses; and QPR is the tangent at P. Two variable parallel tangents to the same conic meet the fixed tangent in Q and R; and from Q and R tangents are drawn to the inner ellipse intersecting in O. Show that the locus of O is a circle, having its centre on the normal at P.

Solution by H. R. GREEER, B.A.; J. DALE; *and others.*

Let Δ be the eccentric angle of the point of contact of the fixed tangent to the exterior ellipse (axes = A, B); let $\alpha, \beta, \gamma, \delta$ be the eccentric angles of the points of contact of the four tangents drawn as directed, in the pairs α and β, γ and δ, to the inner ellipse (axes = a, b). Then the equations of the tangents at $\alpha, \beta, \gamma, \delta$ are respectively

$$\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha = 1 \ldots\ldots (1), \qquad \frac{x}{a}\cos\beta+\frac{y}{b}\sin\beta = 1 \ldots\ldots (2),$$

$$\frac{x}{a}\cos\gamma+\frac{y}{b}\sin\gamma = 1 \ldots\ldots (3), \qquad \frac{x}{a}\cos\delta+\frac{y}{b}\sin\delta = 1 \ldots\ldots (4).$$

(1) and (2) intersect at $\frac{x}{a}=\frac{\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}=\frac{\cos S}{\cos D}$, $\frac{y}{b}=\frac{\sin S}{\cos D}$;

(3) and (4) intersect at $\frac{x}{a}=\frac{\cos\frac{1}{2}(\gamma+\delta)}{\cos\frac{1}{2}(\gamma-\delta)}=\frac{\cos S_1}{\cos D_1}$, $\frac{y}{b}=\frac{\sin S_1}{\cos D_1}$.

The conditions that these points lie on the tangent at P are

$$\frac{a}{A}\cos\Delta\cos S+\frac{b}{B}\sin\Delta\sin S=\cos D\ \ldots\ldots\ (5);$$

$$\frac{a}{A}\cos\Delta\cos S_1+\frac{b}{B}\sin\Delta\sin S_1=\cos D_1\ \ldots\ldots\ (6).$$

Also, if θ be the eccentric angle of the point of contact of the second tangent drawn from the intersection of (1) and (2), the eccentric angle of the second tangent drawn from the intersection of (3) and (4) is $\pi+\theta$; therefore the following conditions also hold good:—

$$\frac{a}{A}\cos\theta\cos S+\frac{b}{B}\sin\theta\sin S=\cos D\ \ldots\ldots (7);$$

$$\frac{a}{A}\cos\theta\cos S_1+\frac{b}{B}\sin\theta\sin S_1=-\cos D_1\ \ldots (8).$$

Eliminating θ and Δ from (5), (6), (7), (8), we have

$$\frac{A^2}{a^2}(\sin S\cos D_1-\sin S_1\cos D)^2+\frac{b^2}{B^2}(\cos S\cos D_1-\cos S_1\cos D)^2$$

$$=\frac{A^2}{a^2}(\sin S\cos D_1+\sin S_1\cos D)^2+\frac{b^2}{B^2}(\cos S\cos D_1+\cos S_1\cos D)^2$$

$$=\sin^2(S-S_1);$$

therefore $$\frac{A^2}{a^2}\sin S\sin S_1+\frac{B^2}{b^2}\cos S\cos S_1=0,$$

and $$A^2=a^2\frac{\cos S\cos S_1\sin(S-S_1)}{\sin S\cos S_1\cos^2 D_1-\sin S_1\cos S_1\cos^2 D},$$

$$-B^2=b^2\frac{\sin S\sin S_1\sin(S-S_1)}{\sin S\cos S\cos^2 D_1-\sin S_1\cos S_1\cos^2 D}.$$

Substituting these values in the condition $A^2-B^2=a^2-b^2$, the result is

$$a^2\left\{\sin(\alpha+\beta)\sin\gamma\sin\delta-\sin(\gamma+\delta)\sin\alpha\sin\beta\right\}$$
$$=b^2\left\{\sin(\alpha+\beta)\cos\gamma\cos\delta-\sin(\gamma+\delta)\cos\alpha\cos\beta\right\}\ \ldots\ldots\ (9).$$

The equation of any conic passing through the intersection of (1) and (2) with (3) and (4) is

$$\left(\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha-1\right)\left(\frac{x}{a}\cos\beta+\frac{y}{b}\sin\beta-1\right)$$
$$+m\left(\frac{x}{a}\cos\gamma+\frac{y}{b}\sin\gamma-1\right)\left(\frac{x}{a}\cos\delta+\frac{y}{b}\sin\delta-1\right)=0.$$

If this conic be a circle, we must have

$$a^2(\sin\alpha\sin\beta+m\sin\gamma\sin\delta)=b^2(\cos\alpha\cos\beta+m\cos\gamma\cos\delta),$$

and $$\sin(\alpha+\beta)+m\sin(\gamma+\delta)=0.$$

Eliminating m, we have the same condition (9) as before. Paying attention to this condition, the equation to the circle becomes

$$a^2\left\{\sin(\gamma+\delta)\sin\alpha\sin\beta-\sin(\alpha+\beta)\cos\gamma\cos\delta\right\}(x^2+y^2)$$
$$+b^2\left\{\sin(\alpha+\beta)(\cos\gamma+\cos\delta)-\sin(\gamma+\delta)(\cos\alpha+\cos\beta)\right\}ax$$
$$+a^2\left\{\sin(\alpha+\beta)(\sin\gamma+\sin\delta)-\sin(\gamma+\delta)(\sin\alpha+\sin\beta)\right\}by$$
$$+a^2b^2\left\{\sin(\gamma+\delta)-\sin(\alpha+\beta)\right\}=0,$$

which may be put in the form

$$(A^2-a^2)(x^2+y^2)+2(Ab^2\cos\Delta)\,x+2(Ba^2\sin\Delta)\,y=A^2b^2+B^2a^2.$$

The coordinates of the centre are

$$x=-\frac{Ab^2\cos\Delta}{A^2-a^2},\quad y=-\frac{Ba^2\sin\Delta}{A^2-a^2},$$

which satisfy the equation to the normal at P,

$$A\sin\Delta\,.\,x-B\cos\Delta-(A^2-B^2)\sin\Delta\cos\Delta=0.$$

The square of the radius

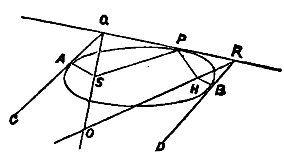
$$=\frac{A^2b^4\cos^2\Delta+B^2a^4\sin^2\Delta+(A^2-a^2)(A^2b^2+B^2a^2)}{(A^2-a^2)^2}$$

$$=\frac{A^2\left\{B^2a^2+b^2(B^2-b^2)\right\}\sin^2\Delta+B^2\left\{A^2b^2+a^2(A^2-a^2)\right\}\cos^2\Delta}{(A^2-a^2)^2}$$

$$=\frac{a^2(A^2-a^2)+b^2(B^2-b^2)+a^2b^2}{(A^2-a^2)}(A^2\sin^2\Delta+B^2\cos^2\Delta);$$

therefore the radius bears a constant ratio to the semi-diameter of the outer ellipse parallel to the fixed tangent.

[Note by the Proposer.—In the particular case where the inner conic reduces to the curve of the second class whose tangents are all the straight lines through the two foci, the locus can be shown geometrically to be a circle. Thus with the notation of the figure, and assuming known properties of the foci:—



$$SOH=AQS+BRH=SAC-ASQ+BRH=HBR-QSP+BRH$$

(since SA is parallel to HB)

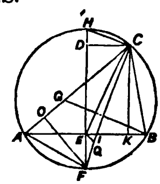
$$=\pi-BHR-QSP=\pi-PHR-QSP=\pi-SOH-SPH$$
$$=2QPS-SOH;$$

therefore $2SOH=2QPS$; therefore $SOH=QPS$, a constant angle; therefore the locus is a circle through S and H.]

3009. (Proposed by F. E. Kitchener, M.A.)—The square on the difference of any two sides of a triangle is equal to four times the rectangle con-

tained by the distances of the middle point of the third side from the points where that side is cut by the perpendicular on it and the bisector of the opposite angle.

I. *Solution by* T. T. WILKINSON, F.R.A.S.



Let ABC be the triangle; E the middle of the base; I where the bisector, and K where the perpendicular, meets the base. Also draw the other lines as in the diagram. Then, if EQ be perpendicular to FC, we have evidently $FQ.FC = FE.FD = FO^2$, by the properties of right-angled and similar triangles.

Now $\quad \frac{1}{2}(AC-BC) = \frac{1}{2}AG = AO;$

and $AO^2 = AF^2 - FO^2 = FE.FH - FE.FD = FE.DH.$

But $\quad FE : EI = CD \text{ (or } EK) : DH;$

therefore $\quad EI.EK = FE.DH = AO^2$

= the square on half the difference of the sides.

Similar properties exist for the other two sides.

NOTE.—This is one of the properties of *Halley's Diagram*, which was partially discussed in the four numbers of the *Student*, published at Liverpool from 1797 to 1800. It there forms Prop. 8, and is due to "*Non Sibi*," a name assumed by the first Editor, Mr. John Knowles. In the diagram, as there considered, the properties of *one* side only are given; but when all the sides are considered, there seems to be no limit to the relations between the different parts of the figure. Some time ago I considered the "angular properties" only; and after writing down about 130 of them, they seemed to arise more abundantly than ever.

II. *Solution by* A. A. BOURNE; *the* REV. T. J. SANDERSON, M.A.; R. W. GENESE; R. TUCKER, M.A.; A. MARTIN; *the* PROPOSER; *and many others.*

By well known theorems, we have

$$AC^2 + CB^2 = 2AE^2 + 2EC^2, \quad \text{and} \quad AC.CB = AI.IB + IC^2;$$

therefore
$$(AC-CB)^2 = 2AE^2 + 2EC^2 - 2AI.IB - 2IC^2$$
$$= 2EC^2 + 2EI^2 - 2IC^2 = 4EI.EK.$$

2700. (Proposed by ARTEMAS MARTIN.)—Three equal coins are piled at random on a horizontal plane; required the probability that the pile will stand.

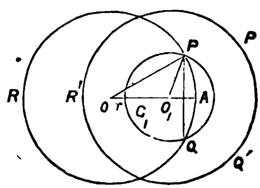
Solution by W. S. B. WOOLHOUSE, F.R.A.S.

The solution of the question will be facilitated, and every possible position of stability will be included, by beginning with the top coin and proceeding downwards. As the top coin is supposed to be placed so as to

touch the second coin radius ρ, its centre must always be contained within the surface of a concentric circle radius 2ρ; but, in order that it may not fall, its centre must rest upon the surface of the supporting coin radius ρ; and the probability that this will happen, being in proportion with the areas, is therefore $\frac{1}{4}$. Let G designate the common centre of gravity of the two coins so placed. Then again the second coin will touch the third coin if the centre of the former fall within a concentric circle radius 2ρ; and accordingly the total positions of the common centre G will range over a like circle radius 2ρ; but, in order that the pile already formed by the first and second coins may not fall, the common centre G must rest on the surface of the third coin, and so be found within a circle radius ρ, the probability of which is again $\frac{1}{4}$. Hence the joint probability that all three coins will remain stable when placed at random is $\frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$.

By similar reasoning it will appear that the probability that an aggregate pile of n equal coins will stand is 4^{1-n}.

The probability just found is complete, and comprises all compounded positions of ultimate stability. If the pile be built up by one coin at a time, commencing from the lowermost coin, and the successive piles be severally required to stand, we shall then obtain a partial probability which takes into account only a part of the foregoing positions. To determine this partial probability, we observe that when the second coin supports the third or uppermost coin, the centre of the latter must rest upon its disc, and therefore that the common centre of gravity G must always be found within a concentric circle radius $\frac{1}{2}\rho$, and that its positions are equally distributed over that circle; also that when the second coin is separately stable, its own centre O_1, as well as the common centre G, must rest upon the disc of the lowermost coin.



In the diagram, the circle PQR, radius $OP = \rho$, represents the lowermost coin; the circle PQr, radius $O_1P = \frac{1}{2}\rho = \rho_1$, the concentric circle on the second coin P′Q′R′ to which the common centre G_1 is necessarily limited; $OO_1 = x$; $PQ = 2y$;

$$\angle POA = \theta; \quad \angle PO_1A = \theta_1;$$

and $\qquad \angle OPO_1 = \theta_1 - \theta = \omega.$

Since the new condition requires the centre O_1 as well as the centre G_1 to be on PQR, the latter centre must be restricted to that portion PAQr of the surface which is common to both circles, viz.,

$$M = \rho_1^2(\pi - \theta_1) + \rho_1^2 \cos\theta_1 \sin\theta_1 + \rho^2\theta - \rho^2 \cos\theta \sin\theta \text{ ...... } (\alpha).$$

But $\qquad x = \rho\cos\theta - \rho_1\cos\theta_1, \quad$ and $\quad y = \rho\sin\theta = \rho_1\sin\theta_1;$

$$\therefore\ dM = 2\rho^2 d\theta \sin^2\theta - 2\rho_1^2 d\theta_1 \sin^2\theta_1 = 2y(\rho\, d\theta \sin\theta - \rho_1 d\theta_1 \sin\theta_1) = -2y\,dx.$$

Also, $\quad x^2 = \rho^2 + \rho_1^2 - 2\rho\rho_1\cos\omega, \quad x\,dx = \rho\rho_1\,d\omega \sin\omega, \quad$ and $\quad xy = \rho\rho_1 \sin\omega.$

Now, as the positions of O_1 are proportional to $x\,dx$, the favourable positions of G_1 are

$$\int x\,dx\,.\,M = \tfrac{1}{2}x^2M - \tfrac{1}{2}\int x^2 dM = \tfrac{1}{2}x^2M + \int x^2 y\,dx = \tfrac{1}{2}x^2M + \rho^2\rho_1^2\int d\omega \sin^2\omega$$
$$= \tfrac{1}{2}x^2M + \tfrac{1}{2}\rho^2\rho_1^2(\omega - \cos\omega\sin\omega) \text{ .... } (\beta).$$

That is, between the limits $x = 0$ and $x = \rho$,

$$\int x\,dx\,.\,\mathrm{M} = \tfrac{1}{2}\rho^2(\rho_1{}^2\pi-\rho_1{}^2\theta_1+\rho^2\theta-xy)+\tfrac{1}{2}\rho^2\rho_1{}^2(\theta_1-\theta-\cos\omega\sin\omega)$$
$$=\tfrac{1}{2}\rho^2\left\{\rho_1{}^2\pi+(\rho^2-\rho_1{}^2)\,\theta-\rho y-\rho_1{}^2\cos\omega\sin\omega\right\}$$
$$=\tfrac{1}{2}\rho^2\left\{\rho_1{}^2\pi+(\rho^2-\rho_1{}^2)\,\theta-\frac{2\rho^2+\rho_1{}^2}{2\rho}\,\rho_1\cos\tfrac{1}{2}\theta\right\},$$

since, when O_1 is at A, $\cos\omega=\sin\frac{1}{2}\theta=\dfrac{\rho_1}{2\rho}$, and $y=\rho_1\sin\omega=\rho_1\cos\frac{1}{2}\theta$.

Also the total positions are $\int x\,dx\,.\,4\rho_1{}^2\pi$ (from $x=0$ to 2ρ) $=8\rho^2\rho_1{}^2\pi$;

therefore $$p=\frac{1}{16}\left(1+\frac{\rho^2-\rho_1{}^2}{\rho_1{}^2\pi}\,\theta-\frac{2\rho^2+\rho_1{}^2}{2\rho\rho_1\pi}\cos\tfrac{1}{2}\theta\right)$$

3 $$=\frac{1}{16}-\frac{3}{16\pi}\left(\frac{9}{16}\sqrt{(15)}-2\sin^{-1}\frac{1}{4}\right)\ \ldots\ldots\ldots\ldots\ldots\ (\gamma),$$

the probability required.

When the aggregate stability of the three pieces is the only condition, then only G_1, and not O_1, is required to rest on the disc PQR; hence, taking the integral (β) between the complete limits $x=0$ and $x=\rho+\rho_1$, we find the favourable positions become $\int x\,dx\,.\,\mathrm{M}=\frac{1}{2}\rho^2\rho_1{}^2\pi$, and thence $p=\frac{1}{16}$, as before shown.

2947. (Proposed by R. Tucker, M.A.)—A is the point on an ellipse through which the osculating circles at three other points B, C, D pass, and (P) is the circle through these four points. If (Q) be the corresponding circle for a point A′, which is the extremity of the diameter conjugate to A, then—(1) the radical axis of (P) and the auxiliary circle (C) touches the ellipse; (2) the radical centre of (P), (Q), and (C) lies on a concentric ellipse whose axes coincide with the original axes.

I. *Solution by the* Rev. J. Wolstenholme, M.A.

If the eccentric angles of B, C, D be $\alpha-\frac{2}{3}\pi$, α, $\alpha+\frac{2}{3}\pi$, that of A will be -3α, and the equation of the circle ABCD is

$$2\,(x^2+y^2)-\left(\frac{x}{a}\cos 3\alpha+\frac{y}{b}\sin 3\alpha\right)(a^2-b^2)-(a^2-b^2)\cos 2\alpha$$
$$-2\,(a^2\sin^2\alpha+b^2\cos^2\alpha)=0;$$

and the equation of the auxiliary circle being $x^2+y^2=a^2$, the radical axis of (P) and (C) is $$\frac{x}{a}\cos 3\alpha+\frac{y}{b}\sin 3\alpha=1,$$

or touches the ellipse at the other extremity of the diameter through A.

Hence the radical centre of (P), (Q), and (C) is the intersection of tangents to the ellipse at the ends of conjugate diameters, and its locus is the ellipse $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2.$$

II. *Solution by* R. W. GENESE.

Let α be the eccentric angle of A, and ϕ that of any of the points B, C, D;

then $\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1$, and $\frac{x}{a}\cos\frac{\phi+\alpha}{2} + \frac{y}{b}\sin\frac{\phi+\alpha}{2} = \cos\frac{\phi-\alpha}{2}$,

i. e., the tangent at ϕ and the chord (ϕ, α) are equally inclined to the axis;

therefore $\tan\phi = -\tan\frac{1}{2}(\phi+\alpha)$, therefore $\phi + \frac{1}{2}(\phi+\alpha) = n\pi$.

This is also readily deduced by Geometry, which shows moreover that the ϕ's of BCD are obtained by putting $n = 1, 2, 3$.

We easily see that $\cos 3\phi = \cos\alpha$ ....(1).

Now let $x^2+y^2+px+qy+r=0$ be the equation to the circle ABCD, its intersections with the ellipse are obtained by putting

$$x = a\cos\theta, \quad y = b\sin\theta,$$

or by
$$a^2\cos^2\theta + b^2\sin^2\theta + pa\cos\theta + qb\sin\theta + r = 0,$$

or by
$$\left\{(a^2-b^2)\cos^2\theta + pa\cos\theta + b^2 + r\right\}^2 = q^2b^2(1-\cos^2\theta).$$

The four roots of this equation are to be the same as $\cos\theta = \cos\alpha$ and $\cos 3\theta = \alpha$, therefore it is identical with

$$(\cos\theta - \cos\alpha)(4\cos^3\theta - 3\cos\theta - \cos\alpha) = 0.$$

Comparing absolute term and coefficient of $\cos^3\theta$, and putting $a^2-b^2=c^2$,

we get $\frac{(b^2+r)^2-q^2b^2}{c^4} = \frac{\cos^2\alpha}{4}$, and $\frac{2pa}{c^2} = -\cos\alpha$.

By interchanging a and b, it is plain that we must have also

$\frac{2qb}{(-c^2)} = -\sin\alpha$; therefore $\frac{(b^2+r)^2}{c^4} = \frac{\sin^2\alpha+\cos^2\alpha}{4}$, and $b^2+r = \pm\frac{1}{2}c^2$.

If we look back to the identical equations, we see that, comparing coefficients of $\cos\theta$, we should get only the upper sign,

therefore $r = -\frac{1}{2}(a^2+b^2)$.

Thus the equation to the circle ABCD is

$$x^2+y^2 - \frac{c^2}{2}\left(\frac{x}{a}\cos\alpha - \frac{y}{b}\sin\alpha\right) - \frac{a^2+b^2}{2} = 0 \text{ ....(P).}$$

The radical axis of this and C, viz. $x^2+y^2=a^2$, is

$$\frac{x}{a}\cos\alpha - \frac{y}{b}\sin\alpha = 1;$$

and this touches the ellipse at the point whose eccentric angle is $2\pi-\alpha$.

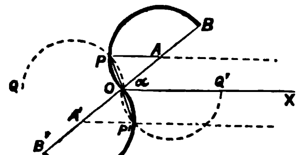
2. The α of Q is $\frac{1}{2}\pi+\alpha$; therefore the corresponding tangents of the auxiliary circle to the above radical axes are at right angles, and intersect on the circle $x^2+y^2=2a^2$.

The radical centre of P, Q, and C therefore lies on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2.$$

2989. (Proposed by the Rev. H. T. SHARPE, M.A.)—A polished wire of small circular section is bent into the form of an S (considered as two semicircles), and is made to rotate rapidly about an axis through its middle point perpendicular to its plane: the sun and the eye being supposed a very long way off in the same plane with the axis of rotation, prove that the appearance due to reflexion will be that of a bright reversed S, thus S.

I. *Solution by* F. D. THOMSON, M.A.



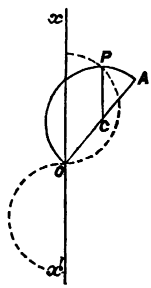
Let OX be the section of the plane of the wire by the vertical plane containing the axis and the sun and eye. Then, since the sun and the eye are supposed to be at an infinite distance, the incident and reflected ray, by which any image of the sun is seen, may be considered as in a vertical plane parallel to OX. This vertical plane must contain the normal at the point of incidence. Now all the normals to any section of the wire are in a plane through the centre of one of the semicircles. Hence, in general, the plane containing the incident and reflected ray, by which the image of the sun is seen, must pass through the centre of one of the semicircles.

The exception is the case where the sun and the eye are on opposite sides of the wire at the same angular elevation, for in this case the *vertical* normal to the wire at *any* point bisects the angle between the two rays.

Hence, if A be the centre of the semicircle OPB, and AP be drawn parallel to OX, P will appear bright to the distant eye.

Let $OP = r$, $OA = a$, $AOX = \alpha$, $XOP = \theta$; then $r = 2a \sin \frac{1}{2}\alpha$. But $\theta = \frac{1}{2}(\pi + \alpha)$; therefore $r = -2a \cos \theta$. This gives for the locus of P the semicircle OPQ corresponding to the semicircle OPB, and there will be a corresponding semicircle on the other side of OX.

II. *Solution by the* Rev. T. J. SANDERSON, M.A.



We need only consider one semicircle of the S-shaped wire, as its optical effect will clearly be repeated by the other.

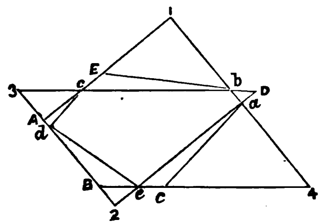
Now, APO being any position of the wire, the bright point of it, as seen by an eye situated as described, will be where a normal plane to the wire is parallel to the plane containing the eye, the sun, and the axis of rotation.

Hence, if xOx' be a section of the latter plane, and, through the centre C of the semicircle, CP be drawn parallel to Ox, P will be the bright point for this position of the wire, so that it only remains to find the *locus* of P.

And the locus of C being a circle, and CP always parallel to itself, but during one half the revolution drawn towards Ox and during the other half towards Ox', it follows that the locus of P will be two semicircles on xx' of the same radius as those of the wire, the one shifted the distance of this radius towards Ox, and the other the same distance towards Ox', forming together a reversed S.

2974. (Proposed by MATTHEW COLLINS, B.A.)—If the polygon *abcde* be interior to ABCDE, and if their corresponding sides be parallel; and if the produced sides of the angle *a* meet the sides of the angle A in the points 1, 2; and if the produced sides of the angle *b* meet the sides of the angle B in 3, 4; &c. &c.; then prove that the polygon 1357 &c. will be equal to the polygon 2468 &c.

I. *Solution by* A. W. RUECKER.



Let the parallelograms A1*a*2, B3*b*4, &c. be called the parallelograms A*a*, B*b*, &c. Then, since AB is parallel to *ab*,

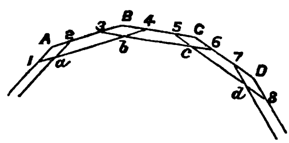
$$\triangle A31 = \triangle A43$$
$$= \tfrac{1}{2}\text{ par. } Bb - \triangle AB4$$
$$= \tfrac{1}{2}\text{ par. } Bb - \tfrac{1}{2}\text{ par. } Aa + \triangle aB2$$
$$= \tfrac{1}{2}\text{ par. } Bb - \tfrac{1}{2}\text{ par. } Aa + \triangle B24;$$

therefore $\triangle A31 + \tfrac{1}{2}\text{ par. } Aa$

$$= \triangle B24 + \tfrac{1}{2}\text{ par. } Bb;$$

and proceeding in the same way for the triangles B53 &c., C46 &c., and adding the equations thus obtained, the semi-parallelograms will disappear; and we find that the excess of the polygons 1357 &c. over ABCD &c. is equal to that of 2468 &c. over ABCD &c.; therefore these polygons are equal.

II. *Solution by* R. W. GENESE.



The area of a polygon is half the moment of the resultant couple of forces represented in position and magnitude by the sides. If, then, we can prove the system of forces 13, 35, 57 ... equivalent to 24, 46, 68 ..., the polygons 1357..., 2468... will be equal. Now a force 13 is equivalent to 12 and a force equal to 23 along *ab*; the first system is therefore equivalent to the series of forces 12, 34, 56, &c. and others equal to 23, 45, 67, &c. along *ab*, *bc*, &c. Again, 24 is equivalent to a force 34 and another equal to 23 along *ab*. Clearly, then, our second system is equivalent to our first, &c.

2985. (Proposed by the Rev. J. BLISSARD.)—To prove that

$$\left\{\frac{\log(1+x)}{x}\right\}^n = 1 - {}_1C_n \cdot \frac{x}{n+1} + {}_2C_{n+1} \cdot \frac{x^2}{(n+1)(n+2)} - \dots$$

where ${}_rC_m$ denotes the sum of the products of the m quantities $1, 2, 3 \dots m$, taken r together.

Solution by the PROPOSER.

The following solution is obtained by use of what I have called Representative Notation, the main principle of which is to employ, under due

and easily understood restriction, super-indices instead of sub-indices in working with any Representative quantity. Thus

$$U_0 + U_1 x + U_2 \frac{x^2}{1.2} + U_3 \frac{x^3}{1.2.3} + \dots \text{ ad inf. is expressed by } U^0 \epsilon^{Ux}.$$

Assume $U^0 \epsilon^{Ux} = x^n$, then expanding we have

$$U_0 + U_1 x + \dots + U_n \frac{x^n}{1.2\dots n} + \dots = x^n;$$

and equating coefficients, all the U quantities, viz. U_0, U_1, U_2, ..., vanish excepting U_n, and we have $\frac{U_n}{1.2\dots n} = 1$ and $U_n = 1.2\dots n$.

Now in $U^0 \epsilon^{Ux} = x^n$, put $\log(1+x)$ for x, and we have

$$U^0 (1+x)^U = \log^n (1+x); \text{ and expanding,}$$

$$U^0 \left(1 + Ux + \frac{U(U-1)}{1.2} x^2 + \dots\right) = x^n \left(1 - \frac{x}{2} + \frac{x^2}{3} + \dots\right)^n.$$

Hence equating coefficients of x^{n+r},

$$\frac{U(U-1)\dots(U-n-r+1)}{1.2\dots(n+r)} = \text{coefficient of } x^r \text{ in } \left(1 - \frac{x}{2} + \frac{x^2}{3} + \dots\right)^n$$

But $U(U-1)\dots\dots(U-n-r+1)$, when expanded, gives

$$U_{n+r} - {}_1C_{n+r-1} \cdot U_{n+r-1} + {}_2C_{n+r-1} \cdot U_{n+r-} - \dots + (-1^r)\, {}_rC_{n+r-1} \cdot U_n \dots,$$

all the terms of which vanish excepting $(-1^r)\, {}_rC_{n+r-1} \cdot U_n$. Hence

$$\frac{(-1^r)\, {}_rC_{n+r-1} \cdot U_n}{1.2\dots(n+r)}, \text{ which } = \frac{(-1^r)\, {}_rC_{n+r-1}}{(n+1)(n+2)\dots(n+r)},$$

$$= \text{coefficient of } x^r \text{ in } \left(1 - \frac{x}{2} + \frac{x^2}{3} + \dots\right)^n.$$

Giving to r the values 1.2.3, &c., we thus have the theorem in the question.

2854. (Proposed by A. MARTIN.)—Solve the equation $x^x = a$, and find the value of x when $a = 300$.

Solution by the PROPOSER.

Taking the Napierian logarithm of both sides of the proposed equation, we have

$$x \log x = \log a = b \dots\dots\dots\dots (1).$$

Putting $(y+1) = \log x$, (2) becomes $x(y+1) = b$ ....(2),

therefore $\log x + \log (y+1) = \log b = c$, or $(y+1) + \log (y+1) = c$ ... (3).

But $\log (y+1) = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \frac{1}{5}y^5 - \frac{1}{6}y^6 + \text{\&c.}$;

therefore $y - \frac{1}{4}y^2 + \frac{1}{6}y^3 - \frac{1}{8}y^4 + \frac{1}{10}y^5 - \frac{1}{12}y^6 + \text{\&c.} = \frac{1}{2}(c-1)$ ....(4).

Reverting this series, we have

$$y = \tfrac{1}{2}(c-1) + \tfrac{1}{16}(c-1)^2 - \tfrac{1}{192}(c-1)^3 - \tfrac{1}{3072}(c-1)^4 + \tfrac{13}{61440}(c-1)^5 \dots;$$

therefore $\log x = 1 + \frac{1}{2}(\log^2 a - 1) + \frac{1}{16}(\log^2 a - 1)^2 - \frac{1}{192}(\log^2 a - 1)^3 \dots$;

and from (2), we have $$x = \frac{\log a}{\log x}.$$

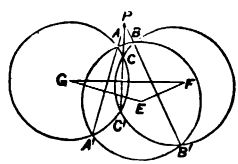
If $a = 300$, we have $\log a = 5{\cdot}703782+$, and $\log^2 a = 1{\cdot}74112$;

therefore $\log x = 1{\cdot}40272+$, and $x = 4{\cdot}0662+$.

2882. (Proposed by M. W. Crofton, F.R.S.)—1. ABC is any equilateral triangle, formed by three arcs of equal circles: if AC, BC be produced to meet in C′, prove that AC′ = BC′ = 60°.

2. Any three equal circles ABB′A′, ACC′A′, BCC′B′ form by their intersections the circular triangles ABC, A′B′C′ (C, C′ being within the circle ABB′A′); prove that the arcs AC + BC − AB = A′C′ + B′C′ − A′B′.

Solution by Asher B. Evans, M.A.



1. Since the chords AC, BC are equal, the arcs AC′, BC′ are equal, and CC′ produced makes an angle of 30° with BC. The arc BC′ is therefore twice the measure of 30°; hence AC′ = BC′ = 60°.

2. Draw the triangle EFG, its vertices being the centres of the circles; then will AA′ be perpendicular to GE, BB′ to EF, CC′ to GF. The angle formed by A′A and C′C, which is measured by ½ (A′C′ − AC), is equal to ∠G; the angle formed by C′C and B′B, which is measured by ½ (B′C′ − BC), is equal to ∠F; and the angle formed by A′A and B′B, which is measured by ½ (A′B′ − AB), is equal to 180° − ∠E. Therefore, since ∠G + ∠F = 180° − ∠E, we have

$$\tfrac{1}{2}(A'B' - AB) = \tfrac{1}{2}(A'C' - AC) + \tfrac{1}{2}(B'C' - BC),$$

or $$AC + BC - AB = A'C' + B'C' - A'B'.$$

Otherwise: The radical centre of the three circles is P, the common intersection of A′A, B′B, C′C. Hence we have

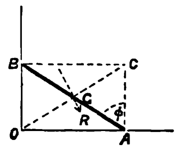
$$2\angle A'PC' = A'C' - AC, \quad 2\angle C'PB' = C'B' - CB, \quad 2\angle A'PB' = A'B' - AB.$$

Hence, since $\angle A'PC' + \angle C'PB' = \angle A'PB'$, we have

$$AC + BC - AB = A'C' + B'C' - A'B'.$$

3041. (Proposed by Professor SYLVESTER.)—A smooth homogeneous beam inclined at 60° to the vertical slips between an upright and a horizontal bar; show (1) that the resultant of the effective moving forces is double the horizontal pressure, and (2) that it cuts the beam in the ratio of 1 : 5.

I. *Solution by* R. W. GENESE.



1. Let AC and BC, a vertical and horizontal through the extremities of the beam, meet in C, which is clearly the instantaneous centre of rotation. The initial acceleration R of the centre of gravity G of the beam is at right angles to CG; and since its direction therefore makes an angle of 60° with BC, we see at once that R is twice the horizontal reaction at B.

2. If ϕ be the angle BA makes with the vertical at any time, the couple round G is $\frac{1}{3}\frac{d^2\phi}{dt^2}$ (taking mass of rod and semi-length unity). But since $\angle$ GCA = GAC = ϕ always, R $=\frac{d^2\phi}{dt^2}$; therefore, compounding the couple and R, we see that the total resultant pressure (=R) acts at a distance from G $=\frac{1}{3}$; and being parallel to R, it meets GB initially at a distance from G $=\frac{1}{3}\sec 60° = \frac{2}{3}$; that is, it divides the whole beam in the ratio of 1 : 5.

II. *Solution by* F. D. THOMSON, M.A.; C. R. RIPPIN, M.A.; *and others.*

1. Let θ be the inclination of the beam, length $2a$, to the horizon at time t. Let R be the vertical, S the horizontal pressure, W the weight. The equations of motion are

$$\frac{W}{g}\cdot\frac{d^2}{dt^2}(a\cos\theta) = S, \quad \frac{W}{g}\cdot\frac{d^2}{dt^2}(a\sin\theta) = R-W,$$

$$\frac{W}{g}\cdot\frac{a^2}{3}\cdot\frac{d^2\theta}{dt^2} = Sa\sin\theta - Ra\cos\theta.$$

Or, *initially*, when $\frac{d\theta}{dt} = 0$, $\theta = \alpha$,

$$-\frac{W}{g}a\sin\alpha\frac{d^2\theta}{dt^2} = S, \quad \frac{W}{g}a\cos\alpha\frac{d^2\theta}{dt^2} = R-W,$$

$$\frac{W}{g}\cdot\frac{a}{3}\cdot\frac{d^2\theta}{dt^2} = S\sin\alpha - R\cos\alpha.$$

Hence $-\frac{W}{g}\cdot a\cdot\frac{d^2\theta}{dt^2} = \frac{W}{g}\cdot\frac{a}{3}\cdot\frac{d^2\theta}{dt^2} + W\cos\alpha$, or $a\frac{d^2\theta}{dt^2} = -\frac{3}{4}g\cos\alpha$.

Hence, initially, $S = \frac{3}{4}W\sin\alpha\cos\alpha$, $R = W(1-\frac{3}{4}\cos^2\alpha)$,
and the square of the resultant of the moving forces

$$= S^2 + (R-W)^2 = \tfrac{9}{16}W^2\cos^2\alpha;$$

therefore the resultant $= \frac{3}{4}W\cos\alpha$.

2. To find the point where the resultant meets the beam, let x be its distance from the extremity B. Then, taking moments about this point,

we have $$Sx \sin\alpha + W(a-x)\cos\alpha = R(2a-x)\cos\alpha,$$

or $$x\left\{S\sin\alpha - (W-R)\cos\alpha\right\} = (2R-W)\,a\cos\alpha,$$

or $$\tfrac{3}{4}x(\sin^2\alpha - \cos^2\alpha) = (1-\tfrac{3}{2}\cos^2\alpha)\,a;$$

therefore $x = \frac{2}{3}a \cdot \dfrac{2-3\cos^2\alpha}{\sin^2\alpha - \cos^2\alpha}$, which gives the point required.

When $\alpha = 30°$, $S = \dfrac{3\sqrt{3}}{16}W$, resultant $= \dfrac{3\sqrt{3}}{8}W$, $x = \frac{1}{3}a$,

which agree with the given results.

2949. (Proposed by J. J. Walker, M.A.)—1. Show that the equation of the circle circumscribing the triangle formed by tangents to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0$ drawn from (x', y'), and their chord of contact, is

$$(b^2x'^2 + a^2y'^2)(x^2+y^2) - b^2(r'^2+c^2)x'x - a^2(r'^2-c^2)y'y + c^2(b^2x'^2 - a^2y'^2) = 0,$$

where $c^2 = a^2 - b^2$, and $r'^2 = x'^2 + y'^2$.

2. Show geometrically that when (x', y') is on one of the equi-conjugat diameters, the circle passes through the centre of the ellipse.

I. *Solution by* James Dale.

1. Combining the equation to the ellipse with the equation to the polar of (x', y'), we get $b^2x^2 + a^2y^2 - b^2x'x - a^2y'y = 0$,

which represents an ellipse passing through (x', y'), through the points of contact of tangents from (x', y'), and having its axes parallel to the given ellipse. The equation to the required circle, then, must be such that, when combined with this equation, the common chords shall be the chord of contact and a line passing through (x', y'), and making the same angle, but in an opposite direction, with the axis as the chord of contact. That is, the equation must be of the form

$$(b^2x^2 + a^2y^2 - b^2x'x - a^2y'y) + m\,(b^2x'x + a^2y'y - a^2b^2)(b^2x'x - a^2y'y - b^2x'^2 + a^2y'^2) = 0.$$

In order that this may represent a circle, we must have

$$m = \frac{a^2-b^2}{b^4x'^2 + a^4y'^2} = \frac{c^2}{b^4x'^2 + a^4y'^2};$$

substituting this value of m, and reducing, we get the equation in the form given above.

2. If P be any point on an equi-conjugate diameter CP (C being the centre), TT′ the polar of P cutting CP in N, it is easily shown that CN . PN = TN . T′N = TN²; therefore the points C, P, T, T′ lie on a circle.

II. *Solution by the* REV. J. WOLSTENHOLME, M.A.; R. TUCKER, M.A.; R. W. GENESE; *and others.*

1. Let $\alpha \pm \beta$ be the excentric angles of tangents drawn from (X, Y) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, so that $X=\frac{a\cos\alpha}{\cos\beta}$, $Y=\frac{b\sin\alpha}{\cos\beta}$; the equation of any circle through the two points of contact is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\left(\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha-\cos\beta\right)\left(\frac{x}{a}\cos\alpha-\frac{y}{b}\sin\alpha-m\right)$$
$$\times\frac{a^2-b^2}{a^2\sin^2\alpha+b^2\cos\alpha}=0.$$

If this pass through (X, Y), we have

$$\frac{\sin^2\beta}{\cos^2\beta}+\frac{\sin^2\beta}{\cos\beta}\left(\frac{\cos 2\alpha}{\cos\beta}-m\right)\left(\frac{a^2-b^2}{a^2\sin^2\alpha+b^2\cos^2\alpha}\right)=0,$$

or $$m\cos\beta=-\frac{a^2\sin^2\alpha+b^2\cos^2\alpha}{a^2-b^2}+\cos 2\alpha=\frac{a^2\cos^2\alpha+b^2\sin^2\alpha}{a^2-b^2};$$

and the equation of the circle becomes

$$(x^2+y^2)\cos\beta-\frac{x}{a}\cos\alpha\,(a^2\cos^2\alpha+b^2\sin^2\alpha+c^2\cos^2\beta)$$
$$-\frac{y}{b}\sin\alpha\,(a^2\cos^2\alpha+b^2\sin^2\alpha-c^2\cos^2\beta)+(a^2-b^2)\cos 2\alpha\cos\beta=0,$$

or $$(x^2+y^2)\left(\frac{X^2}{a^2}+\frac{Y^2}{b^2}\right)-\frac{xX}{a^2}(r^2+c^2)-\frac{yY}{b^2}(r^2-c^2)+c^2\left(\frac{X^2}{a^2}-\frac{Y^2}{b^2}\right)=0.$$

2. If PCP′ be an equi-conjugate, QVQ′ an ordinate to it, and the tangents at Q, Q′ meet in T,

$$QV^2=PV.VP'=CP^2-CV^2=CV.CT-CV^2=CV.VT,$$

or a circle goes round CQTQ′.

Hence, if normals at Q, Q′ meet in O, O will lie on the same circle, and therefore TCO will be a right angle, or O lies on the diameter at right angles to the equi-conjugate.

[Mr. TUCKER remarks, that if (x', y') is on the circle $x^2+y^2=a^2+b^2$, the locus of the centre of the above circle will be

$$(a^2+b^2)(a^2y^2+b^2x^2)^2=a^4b^4(x^2+y^2).]$$

3035. (Proposed by R. W. GENESE.)—OP, OQ are two fixed tangents to a conic; they are met respectively in T and T′ by two variable parallel tangents; prove that OT . OT′ is constant.

I. *Solution by the* Rev. R. TOWNSEND, F.R.S.

For, O being the correspondent in each system to the point at infinity of the other, in the two homographic divisions determined by T and T′ on OP and OQ; therefore, &c.

II. *Solution by* R. TUCKER, M.A.; *the* PROPOSER; *and others.*

Refer the conic to the two tangents as axes, then its equation will be of the form $ax^2+2hxy+by^2+2dx+2ey+f=0$ ....(1), with the conditions $d^2=af$, $e^2=bf$. If we take for the equation to one of the parallel tangents $y=\mu x+\lambda$, we must have, substituting in (1), and remembering that the line is a tangent,

$$\left\{\lambda(h+b\mu)+d+e\mu\right\}^2=(a+2h\mu+b\mu^2)(b\lambda^2+2e\lambda+f),$$

whence $$\lambda^2(h+\sqrt{ab})+2\lambda\sqrt{f}(\sqrt{a}-\mu\sqrt{b})-2\mu f=0;$$

and if λ_1, λ_2 are the values of λ obtained from this equation, and OT, OT′, OT_1, OT_1' the intercepts on the fixed tangent, then

$$\mathrm{OT}\,.\,\mathrm{OT}'=\mathrm{OT}_1\,.\,\mathrm{OT}_1'=\frac{\lambda\lambda'}{\mu}=-\frac{2f}{h+\sqrt{ab}}=\text{a constant}.$$

3051. (Proposed by the Rev. A. F. TORRY, M.A.)—A bright point is placed just within a hollow sphere, and the further hemisphere is polished internally: show that the area of the caustic surface is to that of the sphere as $4\sqrt{2}:45$.

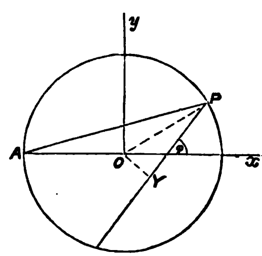
I. *Solution by* SAMUEL ROBERTS, M.A.

We have to compare, between the limits $\theta=(0,\frac{1}{2}\pi)$, the surface (S) formed by the revolution of a Cardioid whose equation is $\rho=\frac{2}{3}(1-\cos\theta)$ with the surface of the sphere whose radius is unity. Now

$$\mathrm{S}=2\pi\int_0^{\frac{1}{2}\pi}\left(\rho^2+\frac{d\rho^2}{d\theta^2}\right)^{\frac{1}{2}}\rho\sin\theta\,d\theta=\frac{8\pi\sqrt{2}}{9}\int_0^{\frac{1}{2}\pi}(1-\cos\theta)^{\frac{3}{2}}\sin\theta\,d\theta$$
$$=\frac{16\pi\sqrt{2}}{45}.$$

Therefore, Σ being the surface of the sphere, $\mathrm{S}:\Sigma=4\sqrt{2}:45$.

II. *Solution by* F. D. THOMSON, M.A.; R. W. GENESE; *and others.*



Let PY be one of the reflected rays, making an angle ϕ with the diameter AO. Then, if OY, the perpendicular on PY from O, be denoted by p, we have $p=a\sin\frac{1}{3}\phi$ as the intrinsic equation to the curve; or, if ρ be the radius of curvature,

$$\rho=\frac{d^2p}{d\phi^2}+p=\frac{8}{9}a\sin\frac{1}{3}\phi;$$

therefore $\delta s=\rho\delta\phi=\frac{8}{9}a\sin\frac{1}{3}\phi\,\delta\phi$.

Now, referred to rectangular axes Ox, Oy, the equation to PY is

$$x\sin\phi-y\cos\phi=a\sin\tfrac{1}{3}\phi\;.............................(1).$$

The point of intersection with the consecutive ray will be found by combining this with $x\cos\phi + y\sin\phi = \frac{1}{3}a\cos\frac{1}{3}\phi$ ....(2).

These give $y = a\left(\frac{1}{3}\sin\phi\cos\frac{1}{3}\phi - \cos\phi\sin\frac{1}{3}\phi\right)$;

therefore the area of the caustic surface is

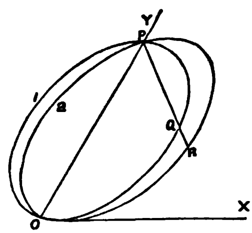
$$\int 2\pi y\delta s = \frac{16\pi a^2}{9}\int_0^{\frac{3}{2}\pi}\left(\tfrac{1}{3}\sin\phi\cos\tfrac{1}{3}\phi - \cos\phi\sin\tfrac{1}{3}\phi\right)\sin\tfrac{1}{3}\phi\, d\phi$$

$$= \frac{16\pi a^2}{3}\int_0^{\frac{1}{2}\pi}\left(\tfrac{1}{3}\sin 3\theta\cos\theta - \cos 3\theta\sin\theta\right)\sin\theta\, d\theta$$

$$= \frac{16\sqrt{2}}{45}\pi a^2 = \frac{4\sqrt{2}}{45}\text{ (surface of sphere).}$$

2836. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—Two conics osculate at O and intersect in P; if any straight line be drawn through P, the locus of the intersection of tangents, drawn to the conics at the points where this line meets them, is a conic touching the former at O, and also touching them again, and the curvature at O of this locus is three-fourths of the curvature of either of the given conics.

I. *Solution by* MATTHEW COLLINS, B.A.



Let OX, the tangent at O, and the line OPY be the axes of x and y; then the equations of the two conics will plainly be of the forms

$$ax^2 + bxy + cy^2 + dy = 0 \quad\text{.........(1)}$$

$$ax^2 + b'xy + cy^2 + dy = 0 \quad\text{.........(2)}$$

since $y = 0$ (axis of x) is plainly a tangent to each at O, and (1)−(2) or $(b-b')\,xy$ indicates that their four intersections lie upon the axes, and $x = 0$ (axis of y) gives $y = 0$ or $-\frac{d}{c}$, hence three of their four points of intersection are at the origin O. The equation of *any* line PQR through their fourth point of intersection P is plainly

$$mx + cy + d = 0 \quad\text{..................................(3).}$$

Let this line cut the curves (1) and (2) in the points Q and R. Now the straight line $bx + cy + d = 0$ ....(4), which plainly passes through P, cuts the conic (1) in two coincident points lying on the line $x = 0$ (axis of y), hence (4) is the tangent at P to (1). Now as the tangents at P and Q may be considered a conic having double contact with (1), we shall have

$$ax^2 + bxy + cy^2 + dy + l\,(mx + cy + d)^2 = (bx + cy + d) \times \text{equation of tan. at Q,}$$

so that the left-hand member must be divisible by $bx + cy + d$ (the quotient, put $= 0$, being the equation of the tangent at Q), and must therefore

vanish if $cy+d$ be taken $=-bx$, *i. e.*, we must have $a+l(m-b)^2=0$; so that we must take $l=-\frac{a}{(m-b)^2}$; and thence the quotient of

$$ax^2+bxy+cy^2+dy-\frac{a}{(m-b)^2}(mx+cy+d)^2 \div (bx+cy+d)$$

gives $$\frac{y}{a}(m-b)^2+(b-2m)x-cy-d=0 \quad\text{......................(5)},$$

which is the equation of the tangent at Q. Hence

$$\frac{y}{a}(m-b')^2+(b'-2m)x-cy-d=0 \quad\text{......................(6)}$$

is the equation of the tangent at R. Now $\{(5)-(6)\}\div(b'-b)$ gives

$$2m=b+b'+\frac{ax}{y},$$

which value, substituted for $2m$ in (5), gives

$$\frac{y}{a}\left(b'-b+\frac{ax}{y}\right)^2=4x\left(b'+\frac{ax}{y}\right)+4(cy+d),$$

or $$2axy(b+b')+3a^2x^2+4ay(cy+d)-y^2(b-b')^2=0 \quad\text{............(7)},$$

the equation of the required locus, which also touches OX at O. Now, near O, where x and y are infinitely small, xy and y^2 are infinitely smaller than y. Rejecting, therefore, these terms in (1), (2), and (7), we then find

from (7), $y=-\frac{3ax^2}{4d}$, and from (1) or (2), $y=-\frac{ax^2}{d}$;

so that the curvature of the required locus at O is three-fourths of the curvature at O of either of the given conics.

Also, since

$$4a(ax^2+bxy+cy^2+dy)-[2axy(b+b')+3a^2x^2+4ay(cy+d)-y^2(b-b')^2]$$
$$=a^2x^2+2axy(b-b')+y^2(b-b')^2=[ax+y(b-b')]^2,$$

therefore the conics (1) and (7) have *double contact* on the chord $ax+y(b-b')=0$, and in like manner (2) and (7) have *double contact* on the chord $ax+(b'-b)y=0$; and hence also these two chords together with OX and OY form a harmonic pencil. [It is obvious, geometrically, that the tangents at P meet (1), (2) in their points of contact with (7).]

II. *Solution by the* Proposer; S. Watson; R. Tucker, M.A.; *and others.*

Take the equations of the two conics

$$x^2+2\lambda xy+by^2-ax=0,\quad x^2+2\mu xy+by^2-ax=0 \quad\text{............(1, 2)},$$

O being the origin, and P on the axis of x. Then a straight line through P, $y=k(x-a)$, meets (1) again in a point, the tangent at which is

$$abk^2(2x+2\lambda y-a)-ka(2\lambda k+1)(2\lambda x+2by)-ax(bk^2+2\lambda k+1)=0 \;\ldots\;\text{(A)}$$

hence if (x, y) be the point of intersection of the two tangents corresponding to this straight line, λ, μ are the roots of the equation (A) considered as a quadratic in λ; therefore

$$\lambda+\mu=\frac{2abk^2y+4kax}{-4k^2ax}=-\frac{by}{2x}-\frac{1}{k};\quad\text{and}\quad \lambda\mu=\frac{abk^2(x-a)-2kaby-ax}{-4k^2ax};$$

whence the equation of the locus is

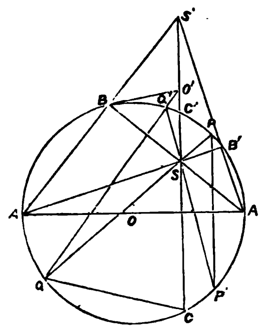
$$\lambda\mu + \frac{b(x-a)}{4x} + \frac{by}{2x}\left(\lambda+\mu+\frac{by}{2x}\right) - \frac{1}{4}\left(\lambda+\mu+\frac{by}{2x}\right)^2 = 0,$$

or
$$\left(1 - \frac{(\lambda-\mu)^2}{b}\right)x^2 + (\lambda+\mu)\,xy + \tfrac{3}{4}by^2 - ax = 0,$$

which satisfies the conditions stated. If this touch the two conics again in R, S, the equations of OR, OS will be found to be $2(\lambda-\mu)\,x \pm by = 0$, or they are harmonically conjugate to OP and the tangent at O.

2856. (Proposed by W. GERAGHTY.)—If two secants SP, SP′ turning round a fixed point S as pole, so as to intercept on a given circle PP′Q′ an arc PP′, whose middle A is a given point, meet the circle again in Q and Q′; prove that the line QQ′ turns round a pole.

Solution by JAMES DALE.



Let AOA′ be a diameter through the given point A; join SA, SA′, meeting the circle again in B, B′; join AB′, A′B, and produce them to meet in S′. Through O′, the middle point of SS′, draw any line meeting the circle in Q, Q′; join QS, Q′S, meeting the circle again in P, P′; then the arc AP is equal to the arc AP′.

Since S is the intersection of the perpendiculars of the triangle AA′S′, SS′ is perpendicular to AA′; and if SS′ cut the circle in C, C′, then the arc AC is equal to the arc AC′. Join O′B, then O′B = O′S, and therefore the angle O′BS = O′SB = AA′B; therefore O′B is a tangent to the given circle. Join CQ and C′Q′; then, because

$$O'Q\,.\,O'Q' = O'B^2 = O'S^2, \text{ therefore } O'Q : O'S = O'S : O'Q';$$

therefore
$$\text{the angle } O'QS = O'SQ',$$

and
$$PQC = O'QC - O'QS = O'C'Q' - O'SQ' = P'Q'C';$$

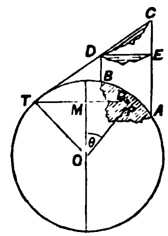
therefore the arc PAC = arc P′AC′; and subtracting each of these equal arcs from CAC′, we get PC′ = P′C, and therefore AP = AP′.

Hence it appears that if chords SPQ, SP′Q′ be drawn through the fixed point S so that the chord PP′ be always perpendicular to the fixed diameter AA′, the chord QQ′ will always pass through the fixed point O′ determined as above. It is evident from the construction that if chords S′pq, S′$p'q'$ be drawn so that pq be always perpendicular to AA′, then $p'q'$ will always pass through O′.

3042. (Proposed by M. W. Crofton, F.R.S.)—1. Let s be any arc of a circle whose plane is vertical. If a tangent be drawn to the circle at a point whose vertical height is the same as that of the centre of gravity of the arc s, the portion of this tangent intercepted between two vertical lines through the extremities of s, is equal to the arc s.

2. Let S represent a portion of the surface of a sphere, the boundary being of any form. Let a cylinder whose sides are vertical pass through this boundary. If any tangent plane be drawn to the sphere, the height of whose point of contact is the same as that of the centre of gravity of the surface S, then the plane area intercepted on this tangent plane by the cylinder is equal to the surface S.

I. *Solution by* F. D. Thomson, M.A.; J. J. Walker, M.A.; *and others.*



Let the figure represent the case of the sphere; and let a be the radius, θ the inclination of OP to the vertical. Then, if δA be an element of area at P, and G be the centre of gravity of the whole area A,

$$\mathrm{OM} = \frac{\Sigma\, \mathrm{OP} \cos\theta \,.\, \delta A}{A}.$$

But the horizontal projection of δA on the plane DE $= \delta A \cos\theta$; therefore

$$\mathrm{OM} = a \,.\, \frac{\text{area DE}}{A}.$$

But area DE = (area DC) cos CDE = (area DC) $\dfrac{\mathrm{OM}}{a}$;

therefore area DC = A.

The same proof applies to the circle.

II. *Solution by* Samuel Roberts, M.A.

1. If r be the radius of the circle and c the chord of the arc in question, ϕ the angle of inclination of the chord to the horizontal diameter, ψ the angle of inclination of the tangent, and t the intercepted length of the same, we have
$$t \cos\psi = c \cos\phi.$$

But $\cos\psi = \dfrac{x}{r}$, where x is the vertical ordinate of the point of contact;

hence
$$t = \frac{rc\cos\phi}{x} = s, \text{ the length of the arc.}$$

2. Analogously, "the distance of the centre of gravity of any portion of the surface of a sphere from the plane of any one of its great circles is a fourth proportional to the area of the portion itself, the area of its projection on this plane, and the radius of the sphere."

Let S be the intercepted surface of the sphere, S′ its projection on a horizontal plane through the centre, r the radius of the sphere; Σ the intercepted portion of the tangent plane, Σ' its projection on the horizontal plane, x the vertical ordinate of the point of contact; then

$$\Sigma' = S', \text{ and } \Sigma = \frac{r}{x}\Sigma' = \frac{r}{x} S' = S.$$

We may generalize this by virtue of the theorem,—"Upon any surface whatever generated by the motion of a sphere of which the centre never departs from a given plane, let any portion S be taken, and let S′ be the projection of S upon the given plane; then the distance of the centre of gravity of S from this plane will be a fourth proportional to S, S′ and the radius of the generating sphere." Hence, instead of a sphere may be taken any surface so generated, the ordinate being the distance from the given plane.

2845. (Proposed by Professor SYLVESTER.)—Find a complete solution of the equation in second differences

$$u_x = u_{x-1} + (x-1)(x-2)\,u_{x-2}.$$

Solution by the PROPOSER.

The answer is $\qquad c\,.\,P_x + k\,.\,Q_x,$

where P_x is the product of x terms of the progression 1, 1, 3, 3, 5, 5, ...,
and Q_x is the product of x terms of the progression 1, 2, 2, 4, 4, 6, 6, ...;

that is, $\quad u_{2x} = c\,.\,1^2.\,3^2.\,5^2 \ldots (2x-1)^2 + k\,.\,2^2.\,4^2 \ldots (2x-2)^2.\,2x,$

and $\quad u_{2x+1} = c\,.\,1^2.\,3^2 \ldots (2x-1)^2(2x+1) + k\,.\,2^2.\,4^2 \ldots (2x)^2.$

I think such a solution of such an equation is quite *sui generis.*

Take $\quad 1^2;\ (1^2.\,3),\ (1^2.\,3^2),\ \ldots\ 1,\ 1,\ 3,\ 9,\ 45,\ 225,\ \ldots\ldots,$

$3 = 1+2\,.\,1,\ \ 9 = 3+6\,.\,1,\ \ 45 = 9+12\,.\,3,\ \ 225 = 45+20\,.\,9,\ \ldots\ldots$

Again, take $\quad 1,\ 2,\ 4,\ 16,\ 64,\ 384,\ \ldots\ldots,$

$4 = 2+2\,.\,1,\ \ 16 = 4+6\,.\,2,\ \ 64 = 16+12\,.\,4,\ \ 384 = 64+20\,.\,16,\ \ldots\ldots$

[Professor SYLVESTER remarks, that this question is connected with, and offered itself in, the Theory of Reducible Cyclodes.]

3021. (Proposed by W. K. CLIFFORD, B.A.)—The three pairs of foci of a sphero-conic are a, a'; b, b'; c, c'; and p is any point on the sphere. Prove the formulæ

$$\sin aa'.\sin bb'.\sin cc' = 8 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$(\sin aa')^{-\frac{2}{3}} + (\sin bb')^{-\frac{2}{3}} + (\sin cc')^{-\frac{2}{3}} = 0 \quad\ldots\ldots\ldots\ldots (2),$$

$$\frac{(\sin pa\,.\,\sin pa')^3}{(\sin aa')^2} = \frac{(\sin pb\,.\,\sin pb')^3}{(\sin bb')^2} = \frac{(\sin pc\,.\,\sin pc')^3}{(\sin cc')^2} \quad\ldots\ldots (3).$$

Solution by the Rev. R. TOWNSEND, M.A., F.R.S.

The equation of any quadric cone, referred to its three principal planes, being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0,$$

those of its three pairs of focal lines AA′, BB′, CC′, in the planes of yz, zx, xy respectively, are

$$\frac{y^2}{b^2-a^2}+\frac{z^2}{c^2-a^2}=0,\quad \frac{z^2}{c^2-b^2}+\frac{x^2}{a^2-b^2}=0,\quad \frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}=0.$$

Hence, if α, β, γ be the three angles which these three pairs of lines make with the axes of y, z, x respectively, we have at once

$$\sin^2\alpha = -\frac{c^2-a^2}{b^2-c^2},\quad \sin^2\beta = -\frac{a^2-b^2}{c^2-a^2},\quad \sin^2\gamma = -\frac{b^2-c^2}{a^2-b^2},$$

$$\cos^2\alpha = -\frac{a^2-b^2}{b^2-c^2},\quad \cos^2\beta = -\frac{b^2-c^2}{c^2-a^2},\quad \cos^2\gamma = -\frac{c^2-a^2}{a^2-b^2};$$

and therefore at once

$$\sin \mathrm{AA}' = \frac{2\,(c^2-a^2)^{\frac{1}{2}}\,(a^2-b^2)^{\frac{1}{2}}}{b^2-c^2},\quad \sin \mathrm{BB}' = \frac{2\,(a^2-b^2)^{\frac{1}{2}}\,(b^2-c^2)^{\frac{1}{2}}}{c^2-a^2},$$

$$\sin \mathrm{CC}' = \frac{2\,(b^2-c^2)^{\frac{1}{2}}\,(c^2-a^2)^{\frac{1}{2}}}{a^2-b^2};$$

from which formulæ (1) and (2) are manifest.

To prove formulæ (3)—If pp', qq', rr' be the three pairs of perpendiculars from any point xyz upon the three pairs of lines AA′, BB′, CC′; then, since evidently

$$\begin{aligned} p^2 &= x^2+(z\cos\alpha-y\sin\alpha)^2, & p'^2 &= x^2+(z\cos\alpha+y\sin\alpha)^2,\\ q^2 &= y^2+(x\cos\beta-z\sin\beta)^2, & q'^2 &= y^2+(x\cos\beta+z\sin\beta)^2,\\ r^2 &= z^2+(y\cos\gamma-x\sin\gamma)^2, & r'^2 &= z^2+(y\cos\gamma+x\sin\gamma)^2, \end{aligned}$$

we have at once from the above values of $\sin\alpha$, $\cos\alpha$, &c.,

$$\begin{aligned} p^2p'^2(b^2-c^2)^2 &= q^2q'^2(c^2-a^2)^2 = r^2r'^2(a^2-b^2)^2\\ &= x^4(b^2-c^2)^2+y^4(c^2-a^2)^2+z^4(a^2-b^2)^2-2y^2z^2(c^2-a^2)(a^2-b^2)\\ &\quad -2z^2x^2(a^2-b^2)(b^2-c^2)-2x^2y^2(b^2-c^2)(c^2-a^2), \end{aligned}$$

from which, since again from the above values of sin AA′, &c.,

$$\begin{aligned} \sin^2\mathrm{AA}'.(b^2-c^2)^3 = \sin^2\mathrm{BB}'.(c^2-a^2)^3 &= \sin^2\mathrm{CC}'.(a^2-b^2)^3\\ &= 4\,(b^2-c^2)(c^2-a^2)(a^2-b^2); \end{aligned}$$

therefore, &c.

2847. (Proposed by M. W. CROFTON, F.R.S.)—Prove that

$$e^{hD^2}.\,e^{-kx^2} = (1+4hk)^{-\frac{1}{2}}e^{-\frac{kx^2}{1+4hk}},\quad \text{where } D = \frac{d}{dx}.$$

Solution by the PROPOSER.

Let $u = e^{hD^2}e^{-kx^2}$; then, differentiating with regard to h,

$$\frac{du}{dh} = e^{hD^2}D^2e^{-kx^2} = e^{hD^2}(4k^2x^2-2k)\,e^{-kx^2};\quad \text{also}\quad \frac{du}{dk} = e^{hD^2}(-x^2e^{-kx^2});$$

we thus obtain the partial differential equation

$$\frac{du}{dh} + 4k^3\frac{du}{dk} + 2ku = 0,$$

the integral of which is $u = k^{-\frac{1}{2}}\phi\,(4h + k^{-1})$.

To determine the arbitrary function ϕ, we remark that, if $h=0$, $u = e^{-kx^2}$,

therefore $\phi\,(k^{-1}) = k^{\frac{1}{2}}e^{-kx^2}$;

therefore $u = k^{-\frac{1}{2}}\left(4h + \frac{1}{k}\right)^{-\frac{1}{2}} e^{-x^2\left(4h+\frac{1}{k}\right)^{-1}} = (1+4hk)^{-\frac{1}{2}} e^{-\frac{kx^2}{1+4hk}}$.

[Another method is to employ Poisson's ingenious transformation (*Traité de Mécanique*, Tom. ii. p. 356), which gives

$$e^{hD^2}\phi\,(x) = \frac{1}{\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty} e^{-\omega^2}\phi\,(x + 2h^{\frac{1}{2}}\omega)\,d\omega.$$

Mr. Crofton remarks, that he has found the above theorem to possess important applications in the Theory of the Errors of Observations. A Solution, by Mr. Walker, has already been given on p. 67 of Vol. XI. of the *Reprint*.]

2979. (Proposed by W. K. Clifford, B.A.)—Two triads of points abc, $\alpha\beta\gamma$ being taken on a line, let the two triads be called *harmonic* of one another when

$$a\alpha\,.\,b\beta\,.\,c\gamma + a\beta\,.\,b\gamma\,.\,c\alpha + a\gamma\,.\,b\alpha\,.\,c\beta + a\gamma\,.\,b\beta\,.\,c\alpha + a\beta\,.\,b\alpha\,.\,c\gamma + a\alpha\,.\,b\gamma\,.\,c\beta = 0,$$

then (1) The envelope of a line cut harmonically by two cubics is of the third class. (The contravariant $\overline{\alpha 1 1'^3}$.)—(2) This line is also cut harmonically by every pair of cubics through the intersections of the first two.—(3) The envelope of a line cut harmonically by a given cubic and by the cubic made up by the polar line and conic of a given point is the mixed concomitant $\overline{\alpha 12}\,.\,\overline{\alpha 13}^2$.—(4) Two cubics having the same inflexions cut harmonically any line whatever.

Solution by J. J. Walker, M.A.

1. If the two triads of points be determined by the two cubics $(a, b, c, d)(x, y)^3$, $(a', b', c', d')(x, y)^3$, it is readily found, by forming the expressions for the segments in terms of the roots, that the condition of "harmonic" division, as defined, is $ad' - a'd - 3(bc' - b'c) = 0$, or $\overline{12}^3$ (Salmon, *Higher Algebra*, p. 121). Hence the condition for two cubic curves, $u = 0$, $v = 0$, cutting a transversal $\alpha x + \beta y + \gamma z$ harmonically would be found by substituting $-\left(\frac{\alpha}{\gamma}x + \frac{\beta}{\gamma}y\right)$ for z in u and v, and operating on the result with $\overline{12}^3$. But the same thing may be done without substitution by operating on uv directly, considering z as a function of x and y determined by $\alpha x + \beta y + \gamma z = 0$. Then in the

operator $\overline{12}$ for $\frac{d}{dx}$ must be substituted $\frac{d}{dx}+\frac{dz}{dx}\frac{d}{dz}$; *i. e.*, $\frac{d}{dx}-\frac{\alpha}{\gamma}\frac{d}{dz}$; and for $\frac{d}{dy}$, $\frac{d}{dy}-\frac{\beta}{\gamma}\frac{d}{dz}$. Making these substitutions, the new operator becomes

$$\alpha\left(\frac{d}{dy_1}\frac{d}{dz_2}-\frac{d}{dz_1}\frac{d}{dy_2}\right)+\beta\left(\frac{d}{dz_1}\frac{d}{dx_2}-\frac{d}{dx_1}\frac{d}{dz_2}\right)+\gamma\left(\frac{d}{dx_1}\frac{d}{dy_2}-\frac{d}{dy_1}\frac{d}{dx_2}\right),$$

i.e., $\overline{\alpha 12}^3$ on uv equated with zero is the required condition; or, considering $\alpha\beta\gamma$ as variable, the envelope of transversals cut harmonically by u, v.

2. It is well known that the symbol $\overline{12}^3$ is a combinantive operator; *i. e.*, that if satisfied for uv it will be satisfied for $u+kv$, $u+k'v$, or for any two cubics passing through the intersections of u and v.

3. More generally, if v break up into any conic u_3 and right line u_2, the envelope of a transversal cut harmonically by the conic and line and by a proper cubic u_1 will be $\overline{\alpha 12}\,.\,\overline{\alpha 13}^2$ on $u_1u_2u_3$. For $\overline{12}^3 u_1 v$ (u_1, v being binary cubics) $=\frac{d^3u_1}{dx^3}\frac{d^3v}{dy^3}-3\frac{d^3u_1}{dx^2dy}\frac{d^3v}{dx\,dy^2}+3\frac{d^3u_1}{dx\,dy^2}\frac{d^3v}{dx^2dy}-\frac{d^3u_1}{dy^3}\frac{d^3v}{dx^3}$;

but if $v=u_2u_3$, $\frac{d^3v}{dy^3}=3\frac{du_2}{dy}\frac{d^2u_3}{dy^2}$, $\frac{d^3v}{dx\,dy^2}=\frac{du_2}{dx}\frac{d^2u_3}{dy^2}+2\frac{du_2}{dy}\frac{d^2u_3}{dx\,dy}$ ......;

and making these substitutions in the general result, it is easily identified (to a factor) with $\left(\frac{d}{dx_1}\frac{d}{dy_2}-\frac{d}{dy_1}\frac{d}{dx_2}\right)\left(\frac{d}{dx_1}\frac{d}{dy_3}-\frac{d}{dy_1}\frac{d}{dx_3}\right)^2 u_1u_2u_3$.

Hence the envelope $\overline{\alpha 12}^3 uv$ reduces in this case to $\overline{\alpha 12}\,.\,\overline{\alpha 13}^2 u_1u_2u_3$.

[By analogous reasoning it would appear that, when v breaks up into three right lines, the envelope becomes $\overline{\alpha 12}\,.\,\overline{\alpha 13}\,.\,\overline{\alpha 14}$ on $u_1u_2u_3u_4$.]

4. If u, v have the same inflexions, either is the Hessian of the other (to a factor). Take for u the canonical form $ax^3+by^3+cz^3+6dxyz$. Then v will be (to a factor) $ad^2x^3+bd^2y^3+cd^2z^3-(2d^3+abc)\,xyz$; comparing which with $a'x^3+b'y^3+c'x^3+6d'xyz$, $bc'-b'c=0$, $ca'-c'a=0$, $ab'-a'b=0$. But $\overline{\alpha 12}^3 uv$ in this case is actually (to a numerical factor)

$$(bc'-b'c)\,\alpha^3+(ca'-c'a)\,\beta^3+(ab'-a'b)\,\gamma^3.$$

Since the coefficients of α^3, β^3, γ^3 separately vanish, the required condition is therefore in this case satisfied for every transversal.

3052. (Proposed by the Rev. W. A. Whitworth, M.A.)—A catapult is formed by fixing the ends of an elastic string (natural length $=2l$) at points A and A′ on a horizontal plane (AA′ $< 2l$). The bullet is placed at the middle point of the string and drawn back at right angles to AA′ along the plane and let go when the string is on the point of breaking (stretched length $=2l'$). Prove that the velocity of the bullet when it leaves the string is independent of the distance AA′ and is to the velocity

it would have acquired in falling through a vertical space $l'-l$ in the subduplicate ratio of the greatest strain the string can bear to the weight of the bullet.

Solution by the REV. R. TOWNSEND, M.A., F.R.S.; C. R. RIPPIN, M.A.; *and others.*

Denoting by v and $2r$ the velocity of the bullet and the length of the cord, in any extended position of the latter; by V the velocity of the former on leaving the cord, and by V′ that due to the action of gravity through the space $(l'-l)$; by W the weight of the bullet, and by W′ the breaking strain of the chord; then since, by known principles,

$$\frac{W}{g} \cdot v\,dv = 2\,\frac{W'}{(l'-l)} \cdot (r-l)\,dr\,;$$

therefore, at once, by integration between the limits l' and l,

$$W \cdot V^2 = W' \cdot 2g\,(l'-l) = W' \cdot V'^2\,;$$

and therefore &c.

If, in place of a single cord, there were n uniform cords, of the common unextended length $2l$, attached to as many pairs of diametrically opposite points on the circumference of a fixed circle, and all drawing the bullet along the axis of the cone of which the circle is the base and itself the vertex; then since, for the same reason,

$$W \cdot V^2 = W' \cdot 2ng\;(l'-l) = W' \cdot V'^2,$$

the above property, consequently, would still be true in both its parts; the velocity V′, in the second part, being that due to the action of gravity through n times the space $(l'-l)$, in place of through once that space, as in the case of a single cord.

2534. (Proposed by the Rev. R. TOWNSEND, F.R.S.)—It is a well known property in Geometry of Two Dimensions, that, when a system of conics have double contact, a variable chord of any one of them cut in a constant anharmonic ratio by any other of them, (a) is cut in constant anharmonic ratios by them all, (b) touches the same one of them in every position, and (c) determines on every one of them two homographic systems of points, of which the two common points and lines of contact are double points and lines.

Show, analogously, in Geometry of Three Dimensions, that when a system of quadrics have quadruple contact, (that is, when they pass through the four sides of a common quadrilateral, real or imaginary, in space,) a variable chord of any one of them cut in constant anharmonic ratios by any two of them, (a) is cut in constant anharmonic ratios by them all, (b) touches the same two of them in every position, and (c) determines on every one of them two homographic systems of points, of which the four common points and planes of contact are double points and planes.

Solution by W. S. McCAY, B.A.

The first and second parts of the Question are true for a system of quadrics having any common curve of intersection.

Consider the chord joining the points xyz, $x'y'z'$ on the quadric $U + kV = 0$: to find the ratio $(l : m)$ in which this chord is cut by $U + k'V = 0$, we get (Salmon's *Geometry of Three Dimensions*, p. 45)

$$l^2 (U + k'V) + 2lm (P + k'Q) + m^2 (U' + k'V') = 0;$$

and the chord will be cut in a constant anharmonic ratio if the ratio of the roots of this quadratic be constant, which will be the case if

$$\frac{(U + k'V)(U' + k'V')}{(P + k'Q)^2} = C,$$

or (since the points we are considering are on $U + kV$) if

$$\frac{P}{(VV')^{\frac{1}{2}}} + k' \frac{Q}{(VV')^{\frac{1}{2}}} = C' \text{ ....(1)};$$

and so for another quadric the anharmonic ratio will always be constant if

$$\frac{P}{(VV')^{\frac{1}{2}}} + k'' \frac{Q}{(VV')^{\frac{1}{2}}} = C'' \text{ ....(2)};$$

hence the variables $\dfrac{P}{(VV')^{\frac{1}{2}}}$ and $\dfrac{Q}{(VV')^{\frac{1}{2}}}$ are completely determined by (1) and (2); so that the anharmonic ratio will be constant for any other quadric of the system .... (*a*).

The chord touches two quadrics of the system. For if it touch $U + \lambda V$,

$$\frac{(U + \lambda V)(U' + \lambda V')}{(P + \lambda Q)^2} = 1, \quad \text{or} \quad (\lambda - \kappa)^2 = \left(\frac{P}{(VV')^{\frac{1}{2}}} + \lambda \frac{Q}{(VV')^{\frac{1}{2}}}\right)^2,$$

which gives two values for λ .... (*b*).

If the quadrics pass through the sides of a common quadrilateral, then

$$U = \alpha\beta, \quad V = \gamma\delta, \quad P = \alpha\beta' + \beta\alpha', \quad Q = \gamma\delta' + \gamma'\delta;$$

and since $\dfrac{P}{(VV')^{\frac{1}{2}}}$ or $\dfrac{P}{(UU')^{\frac{1}{2}}}$ and $\dfrac{Q}{(VV')^{\frac{1}{2}}}$ are constant,

therefore $$\frac{\alpha\beta' + \beta\alpha'}{(\alpha\beta\alpha'\beta')^{\frac{1}{2}}} = C, \quad \text{or} \quad \left(\frac{\alpha\beta'}{\alpha'\beta}\right)^{\frac{1}{2}} + \left(\frac{\alpha'\beta}{\alpha\beta'}\right)^{\frac{1}{2}} = C;$$

therefore $\dfrac{\alpha}{\alpha'} : \dfrac{\beta}{\beta'} = \text{constant}$ ......(1), so too $\dfrac{\gamma}{\gamma'} : \dfrac{\delta}{\delta'} = \text{constant}$ ... (2).

Also, since $$\alpha\beta = \kappa\gamma\delta, \quad \text{and} \quad \alpha'\beta' = \kappa\gamma'\delta',$$

therefore $$\frac{\alpha}{\alpha'} \cdot \frac{\beta}{\beta'} = \frac{\gamma}{\gamma'} \cdot \frac{\delta}{\delta'};$$

therefore, by (1) and (2), $\dfrac{\alpha}{\alpha'} : \dfrac{\gamma}{\gamma'} = \text{constant}$, and $\dfrac{\beta}{\beta'} : \dfrac{\delta}{\delta'} = \text{constant}$;

therefore $$\frac{\alpha}{\alpha'} : \frac{\beta}{\beta'} : \frac{\gamma}{\gamma'} : \frac{\delta}{\delta'} = a : b : c : d.$$

And therefore the two systems of points are homographic .... (*c*).

The last part is obvious.

2939. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—Chords of an ellipse are drawn subtending a right angle at a fixed point O, and O′ is the second focus of the envelope of these chords: prove (1) that CO, CO′ are equally inclined to the axes; and (2) that the major axis of the envelope is

$$\frac{2ab}{a^2+b^2}\left(a^2+b^2-\mathrm{CO}^2\right)^{\frac{1}{2}}.$$

I. *Solution by* A. W. RUECKER.

1. Reciprocate with respect to O. Let (X) and (XY) in the reciprocal figure represent any point X and line XY in the original. Then the envelope of the chords becomes the director circle of the conic into which the ellipse is reciprocated (since the envelope of chords subtending right angles at O becomes the locus of the intersections of tangents containing right angles). Also (O′) is the line perpendicular to and bisecting the perpendicular from O_1 on its polar with respect to the circle, and (C) is the polar of O_1 with respect to the conic. Draw O_1a, O_1b bisecting internally and externally the angle contained by O_1(O′C) and O_1(OC), and let bO_1 meet the conic in m and n. Then $\{bmO_1n\}$ is harmonic; and since $O_1\{b\,(\mathrm{O'C})\,a\,(\mathrm{OC})\}$ is harmonic; therefore also $\{b\,(\mathrm{O'C})\,a\,\infty\}$ is harmonic. Therefore a and b are conjugate points with respect to the conic and correspond to the axes. Therefore CO, CO′ are equally inclined to the axes. Also, since the conic and circle are concentric, therefore in the original figure the polar of O with respect to the ellipse is the directrix of the envelope.

2. Hence, if A and E are the semi-major axis and eccentricity of the envelope, and (x', y') are the coordinates of O, we easily get

$$\mathrm{A}\left(\frac{1}{\mathrm{E}}-\mathrm{E}\right)=-\frac{a^2y'^2+b^2x'^2-a^2b^2}{\sqrt{(a^4y'^2+b^4x'^2)}} \quad\text{and}\quad \mathrm{AE}=\frac{\sqrt{(a^4y'^2+b^4x'^2)}}{a^2+b^2};$$

therefore $$\mathrm{A}^2(a^2+b^2)^2=a^2b^2\left\{a^2+b^2-(x'^2+y'^2)\right\},$$

therefore $$2\mathrm{A}=\frac{2ab}{a^2+b^2}\sqrt{(a^2+b^2-\mathrm{CO}^2)}.$$

II. *Solution by the* PROPOSER; J. DALE; *and others.*

Taking the origin at O, the equation of the conic is

$$\frac{(x+h)^2}{a^2}+\frac{(y+k)^2}{b^2}=1.$$

Any chord $lx+my=1$ will subtend a right angle at O if the two lines

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+2\left(\frac{hx}{a^2}+\frac{ky}{b^2}\right)(lx+my)+\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)(lx+my)^2=0,$$

which join the points of intersection to the origin, are at right angles, that is, if

$$\frac{1}{a^2}+\frac{1}{b^2}+2\left(\frac{hl}{a^2}+\frac{km}{b^2}\right)+(l^2+m^2)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)=0 \;\;\ldots\ldots\;(\mathrm{A}).$$

Hence the equation of two such parallel chords is

$$\left(\frac{1}{a^2}+\frac{1}{b^2}\right)(lx+my)^2+2\left(\frac{hl}{a^2}+\frac{km}{b^2}\right)(lx+my)+\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)(l^2+m^2)=0,$$

and the envelope is

$$\left\{x^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\frac{2hx}{a^2}+\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right\}\left\{y^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\frac{2ky}{b^2}+\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right\}$$
$$=\left\{xy\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\frac{hy}{a^2}+\frac{kx}{b^2}\right\}^2,$$

or $$(x^2+y^2)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+2\left(\frac{hx}{a^2}+\frac{ky}{b^2}\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)$$
$$+\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)^2=\left(\frac{hy}{a^2}-\frac{kx}{b^2}\right)^2,$$

or $$(x^2+y^2)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\left(\frac{hx}{a^2}+\frac{ky}{b^2}+\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)^2$$
$$=\left(\frac{h^2}{a^4}+\frac{k^2}{b^4}\right)(x^2+y^2),$$

or $$(x^2+y^2)\,\frac{a^2+b^2-h^2-k^2}{a^2b^2}=\left(\frac{hx}{a^2}+\frac{ky}{b^2}+\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)^2,$$

a conic, as we know otherwise, of which the fixed point is a focus, and the polar of the fixed point to the given conic the directrix. Its eccentricity e is given by

$$e^2=\left(\frac{h^2}{a^4}+\frac{k^2}{b^4}\right)\frac{a^2b^2}{a^2+b^2-h^2-k^2};$$

or $$1-e^2=\frac{a^2+b^2}{a^2+b^2-h^2-k^2}\left(1-\frac{h^2}{a^2}-\frac{k^2}{b^2}\right).$$

If 2α be the major axis,

$$\frac{\alpha(1-e^2)}{e}=\text{distance of focus from directrix}$$

$$=\frac{1-\dfrac{h^2}{a^2}-\dfrac{k^2}{b^2}}{\left(\dfrac{h^2}{a^4}+\dfrac{k^2}{b^4}\right)^{\frac12}}=\alpha\,.\,\frac{a^2+b^2}{ab\,(a^2+b^2-h^2-k^2)^{\frac12}}\,.\,\frac{1-\dfrac{h^2}{a^2}-\dfrac{k^2}{b^2}}{\left(\dfrac{h^2}{a^4}+\dfrac{k^2}{b^4}\right)^{\frac12}},$$

or $$2\alpha=\frac{2ab}{a^2+b^2}(a^2+b^2-h^2-k^2)^{\frac12}=\frac{2ab}{a^2+b^2}(a^2+b^2-\mathrm{CO}^2)^{\frac12}.$$

The centre is given by

$$\frac{x}{b^2h}=\frac{y}{a^2k}=\frac{\dfrac{hx}{a^2}+\dfrac{ky}{b^2}+\dfrac{h^2}{a^2}+\dfrac{k^2}{b^2}-1}{a^2+b^2-h^2-k^2}=\frac{\dfrac{h^2}{a^2}+\dfrac{k^2}{b^2}-1}{a^2+b^2-h^2-k^2-\dfrac{h^2b^2}{a^2}-\dfrac{k^2a^2}{b^2}}=\frac{-1}{a^2+b^2};$$

and therefore the focus is given by $\dfrac{\mathrm{X}+h}{h}=-\dfrac{\mathrm{Y}+k}{k}=\dfrac{a^2-b^2}{a^2+b^2}$;

whence CO, CO′ are equally inclined to the axis of the ellipse, and OO′ is divided by the axis of the ellipse in the ratio $a^2+b^2:a^2-b^2$.

3058. (Proposed by M. COLLINS, B.A.)—Prove that, in a spherical triangle, $a+b+c=\frac{1}{2}\pi$ and $\tan 2b=2$, if $A=\frac{1}{3}\pi$, $B=\frac{1}{5}\pi$, and $C=\frac{1}{2}\pi$.

I. *Solution by* MORGAN JENKINS, M.A.

For a geometrical proof of both theorems, I will premise that the construction and method of proof of Euclid I. 16 are applicable to a spherical triangle, but that the theorem is to be modified, thus:—

If E be the middle point of AC, the exterior angle $\pi - C$ is $>, =,$ or $<$ A as BE is $>, =,$ or $< \frac{1}{2}\pi$; also BE is $>, =,$ or $< \frac{1}{2}\pi$ as BC + BA is $>, =,$ or $< \pi$; *i.e.*, A + C is $<, =,$ or $> \pi$ as $a+c$ is $<, =,$ or $> \pi$.

In the proposed triangle, produce AC to an equal distance to P, and join PB; draw Pm perpendicular to AB produced; produce Am to an equal distance to n, and join Pn.

Then the triangles ACB, PCB, also the triangles APm, nPm, are, from construction, respectively similar in respect of the magnitude of their sides and angles. So also are the triangles PBC, PBm; for since ABC and PBC each $=\frac{1}{5}\pi$, therefore PBm $=\frac{1}{5}\pi=$ PBC; therefore we have

$$AP = nP = 2b,\quad Am = nm = c+a,\quad APm = nPm = \tfrac{2}{5}\pi,$$

$$APn = \tfrac{4}{5}\pi,\quad \text{and}\quad PAn = \tfrac{1}{5}\pi;$$

therefore $\quad Pn + An = \pi, \quad$ or $\quad 2b+2(c+a)=\pi, \quad$ or $\quad a+b+c=\frac{1}{2}\pi$.

Again, $Am = c+a = \frac{1}{2}\pi - b$; therefore from the right-angled triangle APm, $\cos AP = \cos Am \cos Pm$, or $\cos 2b = \cos(\frac{1}{2}\pi - b)\cos b$; *i.e.*, $\tan 2b = 2$.

[The first part of the theorem is given in the examples of Todhunter's *Spherical Trigonometry*.]

II. *Solution by* R. TUCKER, M.A.; *and others.*

$$\text{Cos } c = \cot A \cot B = \frac{1}{\sqrt{3}}\cdot\frac{\sqrt{5}+1}{\sqrt{(10-2\sqrt{5})}};\quad \therefore\quad \sin^2 c = \frac{4(3-\sqrt{5})}{3(5-\sqrt{5})};$$

whence $\qquad \cot c = \frac{1}{4}(3+\sqrt{5})$ ....(1).

$$\text{Sin }(a+b) = \sin a\cos b + \cos a \sin b = \sin c(\cos A + \cos B)$$
$$= \tfrac{1}{4}(3+\sqrt{5})\sin c = \sin c \cot c, \quad \text{by (1)} = \cos c;$$

therefore $\qquad a+b+c=\frac{1}{2}\pi$.

$$\text{Tan } b = \tan c \cos A = \tfrac{1}{4}(3-\sqrt{5})(\sqrt{5}+1) = \tfrac{1}{2}(\sqrt{5}-1);$$

therefore $\quad \tan 2b = \dfrac{\sqrt{5}-1}{1-\frac{1}{2}(3-\sqrt{5})} = 2, \quad$ and $\quad \tan 2a = \dfrac{2}{\sqrt{5}}$.

3039. (Proposed by G. O. HANLON.)—Prove that

$$\underline{|n} + \frac{\underline{|n+1}}{1} + \frac{\underline{|n+2}}{\underline{|2}} + \frac{\underline{|n+3}}{\underline{|3}} + \text{to } x \text{ terms} = \frac{\underline{|n+x}}{(n+1)\,\underline{|x-1}}.$$

I. *Solution by* Henry Hoskins.

Putting S_n for the sum of n terms, we have

$$S_2 = \underline{\vert n}\left\{1+(n+1)\right\} = \frac{\underline{\vert n+2}}{(n+1)\,1},$$

$$S_3 = S_2 + \frac{\underline{\vert n+2}}{\underline{\vert 2}} = \frac{\underline{\vert n+2}}{n+1}\left\{1+\frac{n+1}{\underline{\vert 2}}\right\} = \frac{\underline{\vert n+3}}{(n+1)\,\underline{\vert 2}},$$

$$S_4 = S_3 + \frac{\underline{\vert n+3}}{\underline{\vert 3}} = \frac{\underline{\vert n+3}}{(n+1)}\left\{\frac{1}{\underline{\vert 2}}+\frac{n+1}{\underline{\vert 3}}\right\} = \frac{\underline{\vert n+4}}{(n+1)\,\underline{\vert 3}}$$

&c. &c. &c.

And, generally, the sum of x terms is $S_x = \dfrac{\underline{\vert n+x}}{(n+1)\,\underline{\vert x-1}}$.

II. *Solution by* J. A. McNeill; J. J. Walker, M.A.; R. Tucker, M.A.; *and others.*

$$\underline{\vert n} + \frac{\underline{\vert n+1}}{1} + \frac{\underline{\vert n+2}}{\underline{\vert 2}} + \frac{\underline{\vert n+3}}{\underline{\vert 3}} + \text{\&c. to } x \text{ terms}$$

$$= \underline{\vert n}\left\{1+\frac{n+1}{1}+\frac{(n+1)(n+2)}{\underline{\vert 2}}+\frac{(n+1)(n+2)(n+3)}{\underline{\vert 3}} + \text{\&c. to } x \text{ terms}\right.$$

$$= \underline{\vert n}\left\{\text{sum of coefficients of the first } x \text{ terms of expansion } (1-y)^{-(n+1)}\right\}$$

$$= \underline{\vert n}\left\{\text{coefficient of } y^{x-1} \text{ in expansion } (1-y)^{-(n+2)}\right\}$$

$$= \underline{\vert n}\left\{\frac{(n+2)(n+3)\ldots\ldots(n+x)}{\underline{\vert x-1}}\right\}$$

$$= \frac{\underline{\vert n}\,(n+1)(n+2)(n+3)\ldots\ldots(n+x)}{(n+1)\,\underline{\vert x-1}} = \frac{\underline{\vert n+x}}{(n+1)\,\underline{\vert x-1}}.$$

Note on Question 3039.

By the Rev. W. A. Whitworth, M.A.

This series is a particular case of the series

$$\frac{\underline{\vert a}}{\underline{\vert b}} + \frac{\underline{\vert a+1}}{\underline{\vert b+1}} + \frac{\underline{\vert a+2}}{\underline{\vert b+2}} + \ldots\ldots$$

of which the sum to n terms is

$$\frac{\dfrac{\lfloor\!\underline{n+a}}{\lfloor\!\underline{n+b-1}} - \dfrac{\lfloor\!\underline{a}}{\lfloor\!\underline{b-1}}}{a-b+1},$$

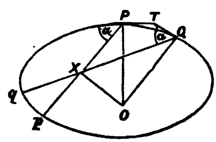
whether a be greater or less than b.

The solution becomes nugatory only in the case when $a-b+1=0$, in which case the series is harmonic.

This solution was given in the *Messenger of Mathematics*, Vol. II., p. 92.

2962. (Proposed by C. Taylor, M.A.)—Prove that a chord of constant inclination to the arc of a closed curve divides the area most unequally when it is a chord of curvature.

I. *Solution by* R. W. Genese.



The area is most unequally divided when the area of either segment is a maximum or a minimum. The geometrical expression for the condition $dA = 0$ is that the area $PXQ = pXq$, where Qq is a consecutive position of the required chord Pp, and X their intersection; and hence that $PX = Xp$.

Let PO, QO be the normals at P, Q; then Pp, Qq make equal angles with them; therefore a circle will go round PXOQ in the limit, therefore OXP is a right angle.

Now O in the limit is the centre of curvature at P, and X is the middle point of Pp; therefore Pp is a chord of curvature.

II. *Solution by the* Rev. J. Wolstenholme, M.A.

If Pp, Qq be two such chords meeting in X, and T be the intersection of tangents at P, Q, a circle goes round XPTQ; but when Q moves up to P, the limiting position of the circle PTQ is a circle touching the curve at P, and of half the linear dimensions of the circle of curvature; hence the limiting position of X is such that PX is half the chord of curvature along Pp. But when the chord divides the oval most unequally, one of the two areas is a maximum and the other a minimum, and the ultimate elementary areas PXQ, pXq must be equal, or Pp must be bisected by the limiting position of X. That is, Pp must be a chord of curvature.

3018. (Proposed by the Rev. J. Wolstenholme, M.A.)—
1. From a fixed point O are drawn tangents OP, OQ to a series of con-

focal conics of which S, S′ are the foci; the envelope of the normals at P, Q will be the parabola which is the well-known envelope of PQ.

2. The circle about OPQ will pass through another fixed point.

3. The conic through OPQSS′ will pass through a fourth fixed point.

4. If a series of conics be inscribed in a fixed quadrilateral of which AA′ is a diagonal, and from a fixed point O tangents OP, OQ be drawn to one of the conics, the conic through OPQAA′ will pass through a fourth fixed point O′, which may be constructed by taking another diagonal BB′, and the pencils A (OBB′O′), A′ (OBB′O′) are both harmonic.

5. If a series of conics pass through four fixed points A, B, C, D, and one of the conics meet a fixed straight line L in P, Q; then the conic touching AB, CD, L and the tangents at P, Q will have a fourth fixed tangent, which with L divides AB and CD harmonically, and is therefore at once to be constructed.

I. *Solution by the* Rev. R. Townsend, M.A., F.R.S.

Of the five different parts of this question, (2) and (3) are evidently contained in (4), of which (5) is the reciprocal; and (4) may be proved immediately as follows:—

4. Let E and E′ on the line PQ be the poles of the two lines OA and OA′ with respect to the conic OAA′PQ; then, since the two lines AE and A′E′ intersect at the point O′, and since the two pencils of rays A (OO′PQ) and A′ (OO′PQ) are both harmonic, the six points A and A′, O and O′, P and Q, lie on the same conic, and therefore, &c.

Let PX and QY be the two lines through P and Q harmonically conjugate to PO and QO with respect to the two triads of angles in involution APA′, BPB′, CPC′, and AQA′, BQB′, CQC′, of which PO and QO are double rays, then, since PX and QY divide homographically the three lines AA′, BB′, CC′, and since, for the two particular conics of the system which pass through O, they coincide with the two tangents OM and ON to those conics, the three lines PX, QY, and PQ consequently envelope the same conic; of this more general property, Part 1 of Question is a well-known particular case.

Note.—If O_A, O_B, O_C be the three-fourth common points for the three systems of conics OAA′PQ, OBB′PQ, OCC′PQ, the three points O_A, O_B, O_C, as lying each on the polar of O with respect to the four sides of the quadrilateral, are collinear.

II. *Solution by the* Proposer.

1. The equation of any conic being $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, (X, Y) the coordinates of O, and θ the eccentric angle of P, the equation of the normal is

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = c^2 \quad \text{.........................(1)},$$

a, b, θ being parameters connected by the equations

$$\frac{X\cos\theta}{a} + \frac{Y\sin\theta}{b} = 1 \quad \text{...........(2)}, \qquad a^2 - b^2 = c^2;$$

whence, for the envelope

$$\left(\frac{ax\sin\theta}{\cos^2\theta}+\frac{by\cos\theta}{\sin^2\theta}\right)d\theta+\frac{x}{\cos\theta}da-\frac{y}{\sin\theta}db=0,$$

$$\left(\frac{\mathrm{X}\sin\theta}{a}-\frac{\mathrm{Y}\cos\theta}{b}\right)d\theta+\frac{\mathrm{X}\cos\theta}{a^2}da+\frac{\mathrm{Y}\sin\theta}{b^2}db=0,\quad\text{and}\quad ada=bdb;$$

whence $$\left(\frac{ax\sin\theta}{\cos^2\theta}+\frac{by\cos\theta}{\sin^2\theta}\right)\left(\frac{\mathrm{X}}{a^3}\cos\theta+\frac{\mathrm{Y}}{b^3}\sin\theta\right)$$

$$=\left(\frac{\mathrm{X}\sin\theta}{a}-\frac{\mathrm{Y}\cos\theta}{b}\right)\left(\frac{x}{a\cos\theta}-\frac{y}{b\sin\theta}\right);$$

or $$\mathrm{X}y\frac{b^2\cos^2\theta}{a^2\sin^2\theta}+\mathrm{Y}x\frac{a^2\sin^2\theta}{b^2\cos^2\theta}=-\mathrm{X}y-\mathrm{Y}x,$$

or $$\frac{\mathrm{X}y}{a^2\sin^2\theta}(a^2\sin^2\theta+b^2\cos^2\theta)+\frac{\mathrm{Y}x}{b^2\cos^2\theta}(a^2\sin^2\theta+b^2\cos^2\theta)=0,$$

or, expelling the factor $a^2\sin^2\theta+b^2\cos^2\theta$, whose geometrical meaning I have not investigated, $\dfrac{\mathrm{X}y}{a^2\sin^2\theta}+\dfrac{\mathrm{Y}x}{b^2\cos^2\theta}=0$ ....(3).

Now, by multiplying (1) and (2), we get

$$\mathrm{X}x-\mathrm{Y}y-c^2=\mathrm{X}y\frac{b\cos\theta}{a\sin\theta}-\mathrm{Y}x\frac{a\sin\theta}{b\cos\theta}$$

and therefore, by (3), $$=2\mathrm{X}y\frac{b\cos\theta}{a\sin\theta}=-2\mathrm{Y}x\frac{a\sin\theta}{b\cos\theta};$$

whence the equation of the envelope is

$$(\mathrm{X}x-\mathrm{Y}y-c^2)^2+4\mathrm{XY}xy=0;$$

or $$\left(x-\frac{c^2\mathrm{X}}{\mathrm{X}^2+\mathrm{Y}^2}\right)^2+\left(y+\frac{c^2\mathrm{Y}}{\mathrm{X}^2+\mathrm{Y}^2}\right)^2=\frac{(x\mathrm{Y}-y\mathrm{X})^2}{\mathrm{X}^2+\mathrm{Y}^2},$$

a parabola whose directrix is (as is otherwise obvious, since the tangents from C to the envelope are the axes, and from O are the bisectors of SOS′) the straight line CO, and whose focus F is such that CF, CO are equally inclined to the axes, and CF . CO = CS^2; also F, O are on the same side of the minor axis.

This, it is well known, is the envelope of PQ, but for the sake of completeness I will find that envelope independently.

The equation of PQ is $\dfrac{x\mathrm{X}}{a^2}+\dfrac{y\mathrm{Y}}{b^2}=1$, where $a^2-b^2=c^2$,

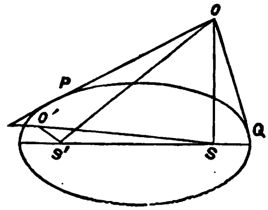
or $$\frac{x\mathrm{X}}{a^2}+\frac{\mathrm{Y}y}{b^2}=\frac{c^2}{a^2-b^2},\quad\text{or}\quad x\mathrm{X}-y\mathrm{Y}-c^2+\frac{a^2}{b^2}y\mathrm{Y}-\frac{b^2}{a^2}x\mathrm{Y}=0;$$

and for the envelope, since this only involves the parameter $\dfrac{b^2}{a^2}$ in the second degree, we have $(x\mathrm{X}-y\mathrm{Y}-c^2)^2+4\mathrm{XY}xy=0$, the same as before.

Hence, if the normals at P, Q meet in R, the circle about PQR passes through the focus F of the envelope; but this is the circle about OPQ, since the angles at P, Q are right angles. Therefore

2. The circle about OPQ passes through another fixed point.

Also, if we now project SS′ into the circular points at infinity, the circular points will be the foci (both real or both impossible) of the new system of conics; and we get (3), which admits of a very simple independent proof.



Draw SO′, S′O′ at right angles to SO, S′O; then, since SO bisects the angle PSQ, SO′ will bisect the external angle, and the pencil S (OPQO′) is harmonic, and so is S′ (OPQO′); or the six points O, P, Q, O′, S, S′ lie on one conic, and PQ, OO′ are conjugates with respect to this conic.

Generalizing by projection, we have (4); and, reciprocating (4), we have (5).

A particular case of (5) is this:—A given straight line meets one of a series of coaxal circles in P, Q; the parabola which touches the given straight line, the radical axis, and the tangents at P, Q, will have another fixed tangent, which may be constructed as follows: take O a point-circle of the system, and let the given straight line meet the radical axis in L, draw OM at right angles to OL to meet the radical axis in M; the fixed tangent passes through M, and is perpendicular to the given straight line. This particular case admits of elementary geometrical proof.

Projecting the original proposition (1), we get—From a fixed point O are drawn tangents OP, OQ to one of a series of conics inscribed in a fixed quadrilateral of which AA′ is a diagonal, OP meets the diagonal in p, and the range (pAA′p') is taken harmonic; prove that the envelope of Pp' is a fixed conic which touches the diagonals of the quadrilateral, and which is also the envelope of PQ.

2711. (Proposed by the Editor.)—To resolve any given number into three rational cubes.

Solution by Samuel Bills; A. Martin; A. B. Evans, M.A.; *and others.*

Let $\dfrac{1+z}{2x}$, $\dfrac{1-z}{2x}$, and $\dfrac{ax^2-1}{x}$ be the roots of the required cubes, and n the given number. Then we must have

$$\frac{(1+z)^3+(1-z)^3}{8x^3}+\frac{(ax^2-1)^3}{x^3}=n,$$

which reduces to $1+3z^2+4a^3x^6-12a^2x^4+12ax^2-4=4nx^3$;

hence $$z^2=1-4ax^2+4a^2x^4-\tfrac{4}{3}a^3x^6+\tfrac{4}{3}nx^3.$$

Assume $$\pm z=1-2ax^2+\tfrac{2}{3}nx^3;$$

then $$z^2=1-4ax^2+4a^2x^4+\tfrac{4}{3}nx^3-\tfrac{8}{3}anx^5+\tfrac{4}{9}n^2x^6.$$

Comparing this value of z^2 with the one above, we have

$$-\tfrac{8}{3}anx^5+\tfrac{4}{9}n^2x^6=-\tfrac{4}{3}a^3x^6;\quad \text{whence}\quad x=\frac{6an}{n^2+3a^3}.$$

To give two or three numerical examples.

1. Let $n=3$, and take $a=3$; then $x=\frac{3}{5}$ and $z=\pm\frac{91}{195}$, and the three roots will be $\frac{17}{75}$, $\frac{36}{25}$, and $\frac{2}{15}$; whence $(\frac{17}{75})^3+(\frac{36}{25})^3+(\frac{2}{15})^3=3$.

2. Let $n=12$, and take $a=4$; then $x=\frac{6}{7}$ and $z=\frac{55}{343}$, therefore
$$(\tfrac{199}{294})^3+(\tfrac{24}{49})^3+(\tfrac{95}{42})^3=12.$$

3. Let $n=15$, and take $a=5$; then $x=\frac{3}{4}$ and $z=\frac{13}{32}$, therefore
$$(\tfrac{15}{16})^3+(\tfrac{19}{48})^3+(\tfrac{29}{12})^3=15.$$

NOTE ON LOGARITHMIC SERIES. *By* ARTEMAS MARTIN.

To develope the Napierian logarithm of $a+x$ into a series.

Differentiating, and then integrating, we have

$$d\left\{\log(a+x)\right\}=\frac{dx}{a+x}=\frac{dx}{a}-\frac{x\,dx}{a^2}+\frac{x^2dx}{a^3}-\frac{x^3dx}{a^4}+\text{\&c.},$$

$$\log(a+x)=\left(\frac{x}{a}\right)-\frac{1}{2}\left(\frac{x}{a}\right)^2+\frac{1}{3}\left(\frac{x}{a}\right)^3-\frac{1}{4}\left(\frac{x}{a}\right)^4+\text{\&c.}+\text{C}.$$

When $x=0$, $\log a=\text{C}$; hence we obtain

$$\log(a+x)=\log a+\left(\frac{x}{a}\right)-\frac{1}{2}\left(\frac{x}{a}\right)^2+\frac{1}{3}\left(\frac{x}{a}\right)^3-\frac{1}{4}\left(\frac{x}{a}\right)^4+\text{\&c.}$$

2709. (Proposed by R. TUCKER, M.A.)—A series of curves being determined by the elimination of θ between

$$\text{X}=\epsilon^{m\theta}(m^2\cos2\theta+2m\sin2\theta+m^2+4),\quad \text{Y}=\epsilon^{m\theta}(m^2\sin2\theta-2m\cos2\theta),$$

where $\text{X}=\dfrac{2x}{c}m(m^2+4)+2(m^2+2)$, and $\text{Y}=\dfrac{2y}{c}m(m^2+4)-2m$;

show that when $m=$ the particular curve will be a cycloid.

Solution by the PROPOSER.

Expanding $\epsilon^{m\theta}$ and neglecting squares of m, we have

$$(1+m\theta)(m^2\cos2\theta+2m\sin2\theta+m^2+4)=\frac{2x}{c}m(m^2+4)+2(m^2+2),$$

$$(1+m\theta)(m^2\sin2\theta-2m\cos2\theta)=\frac{2y}{c}m(m^2+4)-2m;$$

hence, by dividing by m, and then making m vanish, we have

$$\frac{4x}{c}=\sin2\theta+2\theta,\quad\text{and}\quad\frac{4y}{c}=1-\cos2\theta.$$

This gives us, by the elimination of θ,

$$\frac{4x}{c}=\frac{2}{c}\left\{2y(c-2y)\right\}^{\frac{1}{2}}+\sin^{-1}\frac{2}{c}\left\{2y(c-2y)\right\}^{\frac{1}{2}},$$

which is the equation to a cycloid.

3022. (Proposed by the Rev. W. Roberts, M.A.)—Two concentric quadrics, similarly placed, intersect each other. Prove that the cuspidal edge of the developable circumscribed to one of them along the curve of intersection is projected orthogonally, on any of the principal planes, into the evolute of a conic.

Solution by the Rev. R. Townsend, M.A., F.R.S.

The equations of the two quadrics S and S′, referred to their common centre and axes, being respectively

$$ax^2+by^2+cz^2=1 \;\ldots\ldots\ldots\ldots(1), \qquad a'x^2+b'y^2+c'z^2=1 \;\ldots\ldots\ldots\ldots(2),$$

and that of the tangent plane to S at any point (x, y, z) of their curve of intersection being consequently $ax\,.\,\xi+by\,.\,\eta+cz\,.\,\zeta=1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$; differentiating (3) with respect to x, y, z, and eliminating from the result the two ratios $dx : dy : dz$ by means of the differentials of (1) and (2), we get at once the equation

$$\frac{a\,(bc'-b'c)}{x}\,.\,\xi+\frac{b\,(ca'-c'a)}{y}\,.\,\eta+\frac{c\,(ab'-a'b)}{z}\,.\,\zeta=0 \;\ldots\ldots\ldots\ldots(4),$$

which is that of the plane through the centre which intersects the tangent plane (3) in the corresponding edge of the developable in question.

The three parameters x, y, z in the preceding equation (4) being connected by the relation $(a-a')\,x^2+(b-b')\,y^2+(c-c')\,z^2=0 \;\ldots\ldots\ldots\ldots\ldots(5)$, and the three direction angles λ, μ, ν of the plane it represents being consequently connected by the relation

$$l\,.\,\sec^2\lambda+m\,.\,\sec^2\mu+n\,.\,\sec^2\nu=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

where $l : m : n =$ respectively

$$(a-a')\,a^2\,(bc'-b'c)^2 : (b-b')\,b^2\,(ca'-c'a)^2 : (c-c')\,c^2\,(ab'-a'b)^2 \;\ldots\ldots\ldots(7);$$

that plane, therefore, is tangent in every position to the cone

$$l^{\frac{1}{3}}\,.\,x^{\frac{2}{3}}+m^{\frac{1}{3}}\,.\,y^{\frac{2}{3}}+n^{\frac{1}{3}}\,.\,z^{\frac{2}{3}}=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

and normal in every position to the cone

$$\mathrm{A}\,.\,x^2+\mathrm{B}\,.\,y^2+\mathrm{C}\,.\,z^2=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

where $\mathrm{A} : \mathrm{B} : \mathrm{C}$ are given by the relations

$$\mathrm{A}\,(\mathrm{B}-\mathrm{C})^2 : \mathrm{B}\,(\mathrm{C}-\mathrm{A})^2 : \mathrm{C}\,(\mathrm{A}-\mathrm{B})^2 = l : m : n \;\ldots\ldots\ldots\ldots(10).$$

Eliminating successively ξ, η, ζ between the two equations (3) and (4) of the edge of the developable, we obtain, respectively, the three equations

$$\left.\begin{array}{r} b\,(c-c')\,z\,.\,\eta-c\,(b-b')\,y\,.\,\zeta+(bc'-b'c)=0 \\ c\,(a-a')\,x\,.\,\zeta-a\,(c-c')\,z\,.\,\xi+(ca'-c'a)=0 \\ a\,(b-b')\,y\,.\,\xi-b\,(a-a')\,x\,.\,\eta+(ab'-a'b)=0 \end{array}\right\} \;\ldots\ldots\ldots\ldots\ldots(11),$$

which are those of the three projections of the edge upon the three coordinate planes.

The three parameters x, y, z in the three preceding equations (11) being connected, two by two, by the three relations

$$\left.\begin{array}{r} (ab'-a'b)\,y^2+(ac'-a'c)\,z^2=(a-a') \\ (bc'-b'c)\,z^2+(ba'-b'a)\,x^2=(b-b') \\ (ca'-c'a)\,x^2+(cb'-c'b)\,y^2=(c-c') \end{array}\right\} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12);$$

and the three pairs of segments μ_1 and ν_1, ν_2 and λ_2, λ_3 and μ_3, intercepted by the three lines they represent on the axes of y and z, z and x, x and y, respectively, being consequently connected by the three relations

$$m_1 . \mu_1^2 + n_1 . \nu_1^2 = 1, \quad n_2 . \nu_2^2 + l_2 . \lambda_2^2 = 1, \quad l_3 . \lambda_3^2 + m_3 . \mu_3^2 = 1 \quad \text{......(13)},$$

where

$$\left.\begin{array}{ll} m_1 = \left(\dfrac{ab'-a'b}{a-a'}\right)\left(\dfrac{b(c-c')}{bc'-b'c}\right)^2, & n_1 = \left(\dfrac{ac'-a'c}{a-a'}\right)\left(\dfrac{c(b-b')}{bc'-b'c}\right)^2 \\ n_2 = \left(\dfrac{bc'-b'c}{b-b'}\right)\left(\dfrac{c(a-a')}{ca'-c'a}\right)^2, & l_2 = \left(\dfrac{ba'-b'a}{b-b'}\right)\left(\dfrac{a(c-c')}{ca'-c'a}\right)^2 \\ l_3 = \left(\dfrac{ca'-c'a}{c-c'}\right)\left(\dfrac{a(b-b')}{ab'-a'b}\right)^2, & m_3 = \left(\dfrac{cb'-c'b}{c-c'}\right)\left(\dfrac{b(a-a')}{ab'-a'b}\right)^2 \end{array}\right\} \text{...(14)};$$

those three lines are therefore tangent in every position to the three curves

$$m_1^{\frac{1}{3}} . y^{\frac{2}{3}} + n_1^{\frac{1}{3}} . z^{\frac{2}{3}} = 1, \quad n_2^{\frac{1}{3}} . z^{\frac{2}{3}} + l_2^{\frac{1}{3}} . x^{\frac{2}{3}} = 1, \quad l_3^{\frac{1}{3}} . x^{\frac{2}{3}} + m_3^{\frac{1}{3}} . y^{\frac{2}{3}} = 1 \text{ ......(15)},$$

and normal in every position to the three conics

$$\frac{y^2}{m_1} + \frac{z^2}{n_1} = \frac{1}{(m_1-n_1)^2}, \quad \frac{z^2}{n_2} + \frac{x^2}{l_2} = \frac{1}{(n_2-l_2)^2}, \quad \frac{x^2}{l_3} + \frac{y^2}{m_3} = \frac{1}{(l_3-m_3)^2} \text{ ......(16)};$$

and therefore, &c.

When any of the three differences $(bc'-b'c)$, $(ca'-c'a)$, $(ab'-a'b)$ equals 0, the preceding equations containing it break up into factors; and when any two of them, and therefore all three, equal 0, they become all indeterminate. This, it is evident *à priori*, should be the case, the quartic of intersection of the two quadrics S and S′ breaking up in the former case into two conics in planes parallel to the corresponding principal plane of the surfaces, and in the latter case into two conics coinciding in the plane at infinity.

3071. (Proposed by the Rev. W. A. WHITWORTH, M.A.)—A shot is fired in an atmosphere in which the resistance varies as the cube of the velocity. If f be the retardation when the shot is ascending at an inclination α to the horizon, f_0 when it is moving horizontally, and f' when it is descending at an inclination α to the horizon, then

$$\frac{1}{f'} + \frac{1}{f} = \frac{2\cos^3\alpha}{f_0}, \quad \text{and} \quad \frac{1}{f'} - \frac{1}{f} = \frac{2\sin\alpha\,(3-2\sin^2\alpha)}{g}.$$

Solution by the Rev. J. WOLSTENHOLME, M.A.

Let the resistance $= \mu v^n$, the equations of motion will be, ϕ being the inclination of the path to the horizon,

$$\frac{dv}{dt} = -g\sin\phi - \mu v^n, \quad v^2 = -g\frac{ds}{d\phi}\cos\phi, \quad \text{or} \quad v\frac{d\phi}{dt} = -g\cos\phi;$$

therefore $\dfrac{1}{v}\dfrac{dv}{d\phi} = \tan\phi + \dfrac{\mu v^n}{g\cos\phi}$, or $\dfrac{d}{dt}\left(\dfrac{1}{v^n\cos^n\phi}\right) = \dfrac{\mu n}{g(\cos\phi)^{n+1}}$,

an equation giving the retardation in terms of ϕ.

When $n=3$, we have $\dfrac{1}{v^3\cos^3\phi} = C + \dfrac{3\mu}{g}\left(\tan\phi + \dfrac{\tan\phi}{3}\right)$;

whence $$\frac{1}{f} = \frac{\cos^3\alpha}{f_0} + \frac{1}{g}(3\sin^2\alpha\cos^2\alpha + \sin^3\alpha),$$

and $$\frac{1}{f'} = \frac{\cos^3\alpha}{f_0} - \frac{1}{g}(3\sin^2\alpha\cos^2\alpha + \sin^3\alpha),$$

or $$\frac{1}{f} + \frac{1}{f'} = \frac{2\cos^3\alpha}{f_0}, \quad \frac{1}{f} - \frac{1}{f'} = \frac{2\sin\alpha(3-2\sin^2\alpha)}{g}.$$

It is obvious that the equation $\dfrac{1}{f} + \dfrac{1}{f'} = \dfrac{2\cos^n\alpha}{f_0}$ will always be true,

and that $$\frac{1}{f} - \frac{1}{f'} = \frac{2n\cos^n\alpha}{g}\int_0^\alpha \frac{d\phi}{(\cos\phi)^{n+1}};$$

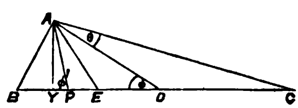
and, if n be an odd integer, the value of this last expression is

$$\frac{2n\cos^n\alpha}{g}\left\{\tan\alpha + \frac{n-1}{2}\,\frac{\tan^3\alpha}{3} + \frac{(n-1)(n-3)}{2.4}\,\frac{\tan^5\alpha}{5} + \ldots \text{to } \frac{n+1}{2} \text{ terms}\right\}.$$

3060. (Proposed by J. J. WALKER, M.A.)—Let ABC be any plane scalene triangle, the side AC being greater than BC. Let the bisector of the base BC, and a second line drawn from A to meet the base, and making the same angle with AB that the bisector makes with AC, be inclined to the base at angles (measured on the side of B) ϕ, ϕ'. Prove that

$$\frac{\cos\phi'}{\cos\phi} = \cos A, \quad \tan\tfrac{1}{2}(\phi'-\phi) = \frac{b-c}{b+c}\tan\tfrac{1}{2}A, \quad \tan\phi' = \frac{b^2+c^2}{b^2-c^2}\tan A.$$

Solution by C. R. RIPPIN, M.A.; R. W. GENESE; R. TUCKER, M.A.; *and others.*



1. Let $\angle ADB = \phi$, $APB = \phi'$, $CAD = BAP = \theta$, $AD = x$, $AP = y$. Draw AE bisecting the angle A, and AY perpendicular to BC; then

$$EP : ED = y : x;$$

and from the equations

$$PB : c = \sin\theta : \sin(\theta + B) \quad \text{and} \quad DC : b = \sin\theta : \sin(C+\theta)$$

we find $PB = \dfrac{ac^2}{b^2+c^2}$; also $EB = \dfrac{ac}{b+c}$, and $EC = \dfrac{ab}{b+c}$;

therefore $EP = \dfrac{abc(b-c)}{(b+c)(b^2+c^2)}$, $ED = \dfrac{a(b-c)}{2(b+c)}$, and $\dfrac{y}{x} = \dfrac{2bc}{b^2+c^2}$.

Also $$\mathrm{DY} = \tfrac{1}{2}a - c\cos \mathrm{B} = \frac{b^2-c^2}{2a},$$

and $$\mathrm{PY} = \frac{ac^2}{b^2+c^2} - c\cos \mathrm{B} = \frac{b^2-c^2}{2a} \cdot \frac{b^2+c^2-a^2}{b^2+c^2};$$

therefore $$\frac{\cos\phi'}{\cos\phi} = \frac{\mathrm{PY}}{\mathrm{DY}} \cdot \frac{x}{y} = \frac{b^2+c^2-a^2}{2bc} = \cos \mathrm{A}.$$

2. $\phi = \theta + \mathrm{C}$, and $\phi' = \pi - (\mathrm{B}+\mathrm{C})$; therefore $\phi + \phi' = \pi - (\mathrm{B}-\mathrm{C})$.

$$\frac{\sin\phi'}{\sin\phi} = \frac{x}{y} = \frac{b^2+c^2}{2bc};$$

therefore $$\frac{\tan\frac{1}{2}(\phi'-\phi)}{\tan\frac{1}{2}(\phi'+\phi)} = \left(\frac{b-c}{b+c}\right)^2 = \frac{b-c}{b+c} \cdot \frac{\tan\frac{1}{2}(\mathrm{B}-\mathrm{C})}{\tan\frac{1}{2}(\mathrm{B}+\mathrm{C})};$$

but $$\tan\tfrac{1}{2}(\phi'+\phi) = \cot\tfrac{1}{2}(\mathrm{B}-\mathrm{C}),$$

therefore $$\tan\tfrac{1}{2}(\phi'-\phi) = \frac{b-c}{b+c}\tan\tfrac{1}{2}\mathrm{A}.$$

3. $$\operatorname{Tan}\phi' = \frac{\mathrm{AY}}{\mathrm{PY}} = \frac{\mathrm{AY}\,.\,2a\,(b^2+c^2)}{(b^2-c^2)(b^2+c^2-a^2)} = \frac{4\text{ Area }(b^2+c^2)}{(b^2-c^2)\,2bc\cos\mathrm{A}} = \frac{b^2+c^2}{b^2-c^2}\,.\tan\mathrm{A}.$$

2851. (Proposed by F. D. Thomson, M.A.)—From a given point a straight line is drawn to meet the tangent to a given conic, so that the two straight lines may be conjugate with respect to a second given conic. The locus of the point of intersection is a quartic curve, whose equation may be expressed in the form $\mathrm{U}^2+\mathrm{V}^2+\mathrm{W}^2 = 0$, where U, V, W are certain conics.

Solution by the Proposer.

1. Refer the two given conics to their self-conjugate triad, then their tangential equations will be of the forms

$$\mathrm{A}\lambda^2+\mathrm{B}\mu^2+\mathrm{C}\nu^2 = 0 \quad\text{and}\quad \mathrm{A}'\lambda^2+\mathrm{B}'\mu^2+\mathrm{C}'\nu^2 = 0 \;\ldots\ldots\ldots\; (1, 2).$$

Let $\lambda x+\mu y+\nu z = 0$, $\lambda' x+\mu' y+\nu' z = 0$ be the trilinear equations to the two straight lines; (x', y', z') the coordinates of the given point.

Then we have to eliminate $\lambda, \mu, \nu, \lambda', \mu', \nu'$ from the equations (1) and

$$\lambda' x' + \mu' y' + \nu' z' = 0 \;\ldots\ldots\ldots\; (3), \qquad \lambda x+\mu y+\nu z = 0 \;\ldots\ldots\ldots\; (5),$$
$$\mathrm{A}'\lambda\lambda' + \mathrm{B}'\mu\mu' + \mathrm{C}'\nu\nu' = 0 \;\ldots\ldots\ldots\; (4), \qquad \lambda' x+\mu' y+\nu' z = 0 \;\ldots\ldots\ldots\; (6).$$

From (3), (4), and (6) we have

$$\mathrm{A}'\lambda\,(y'z-z'y)+\mathrm{B}'\mu\,(z'x-x'z)+\mathrm{C}'\nu\,(x'y-y'x) = 0 \;\ldots\ldots\ldots\ldots (7);$$

therefore, from (5) and (7), we have

$$\lambda : \mu : \nu = \mathrm{C}'y\,(x'y-y'x) - \mathrm{B}'z\,(z'x-x'z) : \&\text{c.} : \&\text{c.},$$

and therefore, from (1), the equation to the locus is

$$A\left\{C'y(x'y-y'x)-B'z(z'x-x'z)\right\}^2+\text{two similar expressions}=0,$$

or, writing for brevity $A'(y'z-z'y) : B'(z'x-x'z) : C'(x'y-y'x) = u : v : w$,

$$A(yw-zv)^2+B(zu-xw)^2+C(xv-yu)^2=0, \quad \text{or} \quad U^2+V^2+W^2=0,$$

where U, V, W are conics passing through (x', y', z') and one of the angles of the self-conjugate triad, and intersecting two and two upon its sides.

2. This proposition, using Professor Cayley's extended sense of the word *perpendicular*, is equivalent to finding the locus of the foot of the perpendicular from a given point on a tangent to a conic.

To derive this particular case, (2) becomes the circular points at infinity, and we may take $C'=0$, $A'=B'=1$, and put $z=1$ throughout. This gives

$$\frac{u}{y'-y}=\frac{v}{x-x'}=\frac{w}{0},$$

and the locus is $A(x-x')^2+B(y'-y)^2+C\left\{x(x-x')-y(y'-y)\right\}^2=0$,

or $$A(x-x')^2+B(y'-y)^2+C\left\{x^2+y^2-xx'-yy'\right\}^2=0,$$

or $$a^2(x-x')^2+b^2(y-y')^2=\left\{x^2+y^2-xx'-yy'\right\}^2 \quad \ldots\ldots\ldots\ldots (8),$$

if the Cartesian equation to the given conic be $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

When (x', y') is at the focus $(ae, 0)$, (8) reduces to

$$(x^2+y^2-a^2)\left\{(x-ae)^2+y^2\right\}=0,$$

which gives the auxiliary circle and the two impossible tangents through the focus.

3. The same work as in (1), interchanging trilinear and tangential coordinates, proves the reciprocal proposition, viz., "P is a point on a conic S, its polar with respect to S′ meets a fixed line in Q. The envelope of PQ is a curve of the fourth class whose equation may be written $U^2+V^2+W^2=0$, where U, V, W are conics touching the given line and a side of the self-conjugate triad, and having two and two common tangents passing through the angles of the conjugate triad."

3068. (Proposed by S. ROBERTS, M.A.)—In the theory of Quintics, the covariant J^2-3K frequently occurs, and if this covariant vanishes the quintic is immediately soluble. What is the meaning of this condition?

Solution by the PROPOSER.

If we suppose the quintic to be of the form $ax^5+5bx^4y+5exy^4$, J^2+lK reduces to $$81b^4e^4+l(27b^4e^4-2ae^5).$$

Now if $ax^4+5bx^3y+5ey^4=0$ represents an equi-anharmonic system, we

have $a=0$ or $e=0$. Hence, if $l=-3$, the vanishing of J^2-3K indicates in this case that four of the roots of the quintic are equi-anharmonic. But in the general case the condition for equi-anharmonicism is evidently invariantive and of the order 8. Hence $J^2-3K=0$ is the condition that some four of the roots of a quintic may be equi-anharmonic.

3067. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—

If $$a^3+b^3+c^3=(c+b)(b+a)(a+c) \qquad (1),$$
and $$(c^2+b^2-a^2)\,x=(a^2+c^2-b^2)\,y=(b^2+a^2-c^2)\,z \qquad (2),$$
then $$x^3+y^3+z^3=(z+y)(y+x)(x+z) \qquad (3).$$

I. *Solution by* C. W. MERRIFIELD, F.R.S.; J. MCNEILL; R. TUCKER, M.A.; *and others.*

The first equation gives
$$a(c^2+b^2-a^2)+b(a^2+c^2-b^2)+c(b^2+a^2-c^2)+2abc=0.$$
Making each term of the second line $=t$, we get for (1)
$$\frac{at}{x}+\frac{bt}{y}+\frac{ct}{z}+2abc=0 \qquad (4).$$
Again, $2c^2=\dfrac{t}{x}+\dfrac{t}{y}=\dfrac{t}{xy}(x+y)$, therefore $c=\left(\dfrac{t(x+y)}{2xy}\right)^{\frac{1}{2}}$,

and so symmetrically. Substituting these in (4), we have
$$\left(\frac{x+y}{z}\right)^{\frac{1}{2}}+\left(\frac{y+z}{x}\right)^{\frac{1}{2}}+\left(\frac{z+x}{y}\right)^{\frac{1}{2}}+\left(\frac{(x+y)(y+z)(z+x)}{xyz}\right)^{\frac{1}{2}}=0.$$
Writing this as $\sqrt{p}+\sqrt{q}+\sqrt{r}+\sqrt{(pqr)}=0$, we get, on transposing and squaring twice,
$$p^2q^2r^2-2pqr(p+q+r)+8pqr+p^2+q^2+r^2=2(qr+rp+pq),$$
or $$\{pqr-(p+q+r)\}^2=4(qr+rp+pq-2pqr).$$
But $pqr-(p+q+r)=2$ (by substitution of $\dfrac{x+y}{z}$ for r, &c.),

and $$qr+rp+pq-2pqr=\frac{1}{xyz}\left\{x^3+y^3+z^3-(y^2z+yz^2+\ldots.)-xyz\right\};$$
therefore $x^3+y^3+z^3=(y^2z+yz^2+\ldots.)+2xyz=(y+z)(z+x)(x+y).$

II. *Solution by the* PROPOSER.

The given relation between a, b, c is equivalent to
$$4abc+(b+c-a)(c+a-b)(a+b-c)=0;$$
therefore $$a^2-(b-c)^2=-\frac{4abc}{b+c-a};$$

therefore $b^2+c^2-a^2 = 2bc\left(1+\frac{2a}{b+c-a}\right) = 2bc\left(\frac{a+b+c}{b+c-a}\right);$

therefore $\frac{x}{a(b+c-a)} = \frac{y}{b(c+a-b)} = \frac{z}{c(a+b-c)} = \frac{y+z-x}{a^2-(b-c)^2} = \ldots = \ldots;$

therefore $\frac{xyz}{(y+z-x)(z+x-y)(x+y-z)} = \frac{abc}{(b+c-a)(c+a-b)(a+b-c)} = -\frac{1}{4},$

from which the given result immediately follows.

3064. (Proposed by the Rev. R. TOWNSEND, M.A., F.R.S.)—If O be the centre and OR the radius of a sphere, real or imaginary, prove the following analogous formulæ in geometry of one, two, and three dimensions respectively, viz.:—

1. If (AB) be the length of the segment determined by any two points A and B in the same line with O, and (A′B′) that of its polar segment with respect to the sphere, then

$$(\mathrm{A'B'}) = \left(\frac{\mathrm{OR}}{1}\right)^2 \cdot \frac{(\mathrm{AB})}{(\mathrm{OA}).(\mathrm{OB})}.$$

2. If (ABC) be the area of the triangle determined by any three points A, B, C in the same plane with O, and (A′B′C′) that of its polar triangle with respect to the same sphere; then

$$(\mathrm{A'B'C'}) = \left(\frac{\mathrm{OR}^2}{1.2}\right)^2 \cdot \frac{(\mathrm{ABC})^2}{(\mathrm{OBC}).(\mathrm{OCA}).(\mathrm{OAB})}.$$

3. If (ABCD) be the volume of the tetrahedron determined by any four points A, B, C, D in the same space with O, and (A′B′C′D′) that of its polar tetrahedron with respect to the sphere; then

$$(\mathrm{A'B'C'D'}) = \left(\frac{\mathrm{OR}^3}{1.2.3}\right)^2 \cdot \frac{(\mathrm{ABCD})^3}{(\mathrm{OBCD}).(\mathrm{OCDA}).(\mathrm{ODAB}).(\mathrm{OABC})}.$$

I. *Solution by* J. J. WALKER, M.A.

1. $$(\mathrm{A'B'}) = (\mathrm{OB'})-(\mathrm{OA'}) = \frac{(\mathrm{OR})^2}{(\mathrm{OB})} - \frac{(\mathrm{OR})^2}{(\mathrm{OA})} = \left(\frac{\mathrm{OR}}{1}\right)^2 \cdot \frac{(\mathrm{OA})-(\mathrm{OB})}{(\mathrm{OA}).(\mathrm{OB})}$$
$$= \left(\frac{\mathrm{OR}}{1}\right)^2 \cdot \frac{(\mathrm{BA})}{(\mathrm{OA}).(\mathrm{OB})}.$$

2. Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be the points A, B, C respectively, referred to any two rectangular radii of the sphere in their plane; then the equations to their polar lines will be $[r = (\mathrm{OR})]\; x_1x+y_1y-r^2 = 0$ ......, and the area of the triangle formed by these three lines is equal to

$$r^4 \times \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}^2 \div 2\,(x_2y_3-x_3y_2)\,(x_3y_1-x_1y_3)\,(x_1y_2-x_2y_1) \ldots\ldots$$

(Salmon's *Conic Sections*, p. 32). The determinant is equal to 2 (ABC), and the divisor to 16 (OBC) . (OCA) . (OAB) ; therefore, &c.

3. Let (x_1, y_1, z_1) ...... $(\ldots z_4)$ be the four points A, B, C, D referred to any three rectangular radii of the sphere; then their polar planes will be $x_1x + y_1y + z_1z - r^2 = 0$......, and the tetrahedron formed by these planes will have its volume equal to

$$r^6 \times \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}^3 \div 6\,(x_2y_3z_4)\,(x_1y_3z_4)\,(x_1y_2z_4)\,(x_1y_2z_3)\ldots\ldots$$

(Salmon's *Geometry of Three Dimensions*, § 35). But the determinant is equal to 6 (ABCD), and the divisor to $(6)^3$. (OBCD) . (OCDA) . (ODAB) . (OABC); therefore, &c.

II. *Solution by the* Proposer.

Of these three analogous formulæ, the first is evident, the second is given in my *Modern Geometry*, Art. 180, and the third may be proved nearly similarly as follows:—

Denoting by X, Y, Z, W the four points inverse to A′, B′, C′, D′ with respect to the sphere, that is, the feet of the four perpendiculars from O upon the four faces BCD, CDA, DAB, ABC of the tetrahedron ABCD; by θ the angle between the two lines OB′ and OC′, that is, between the two planes ABD and ACD; and by ϕ the angle between the line OD′ and the plane B′OC′, that is, between the plane BAC and the line AD: then, for the first of the four tetrahedra OB′C′D′, OC′D′A′, OD′A′B′, OA′B′C′,

since $$(\mathrm{OB'C'D'}) = \tfrac{1}{6} \,.\, (\mathrm{OB'}) \,.\, (\mathrm{OC'}) \,.\, (\mathrm{OD'}) \,.\, \sin\theta \,.\, \sin\phi,$$

substituting for OB′, OC′, OD′ their equivalents $\mathrm{OR}^2 \div \mathrm{OY}$, $\mathrm{OR}^2 \div \mathrm{OZ}$, $\mathrm{OR}^2 \div \mathrm{OW}$; for $\sin\theta$ and $\sin\phi$ their equivalents 3 . (AD) . (ABCD) ÷ 2 . (ABD) . (ACD) and 3 . (ABCD) ÷ (AD) . (ABC); and for (OY) . (ACD), (OZ) . (ABD), (OW) . (ABC) their equivalents 3 . (OACD), 3 . (OABD), 3 . (OABC), we have

$$(\mathrm{OB'C'D'}) = \frac{\mathrm{OR}^6}{36} \,.\, \frac{(\mathrm{ABCD})^2}{(\mathrm{OABC}) \,.\, (\mathrm{OABD}) \,.\, (\mathrm{OACD})},$$

with, of course, similar values for (OC′D′A′), (OD′A′B′), and (OA′B′C′); from which, since identically

$$(\mathrm{OBCD}) + (\mathrm{OCDA}) + (\mathrm{ODAB}) + (\mathrm{OABC}) = (\mathrm{ABCD}),$$

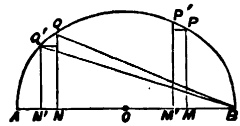
and $$(\mathrm{OB'C'A'}) + (\mathrm{OC'D'A'}) + (\mathrm{OD'A'B'}) + (\mathrm{OA'B'C'}) = (\mathrm{A'B'C'D'});$$

therefore, &c.

3036. (Proposed by R. Tucker, M.A.)—PM is an ordinate to a semicircle (diameter AOB), and PQ is drawn making the same angle with the tangent at P as BP; construct the quadrilateral AQPB when it is a maximum.

I. *Solution by* R. W. Genese.

By the construction, P is the middle point of arc BQ. Thus $\Delta OAQ + 2\Delta OPB$ is to be a maximum.



If P′, Q′ be consecutive positions of P, Q at a maximum, we have momentarily

$\Delta OAQ' + 2\Delta OP'B = \Delta OAQ + 2\Delta OPB$;

therefore, if PM, QN, &c. be the ordinates,

$$Q'N' + 2P'M' = QN + 2PM,$$

i. e., increment of QN = 2 increment of PN. But QQ′ = 2PP′; therefore QQ′ and PP′ are equally inclined to the ordinates;

therefore $\qquad BOP = 60°, \quad BOQ = 120°$;

or the quadrilateral must be half a regular hexagon.

II. *Solution by* J. A. McNeill; R. Tucker, M.A.; *and others.*

Since PQ and PB make equal angles with the tangent at P, the angles PBQ, PQB, PAQ, PAB are each equal to $\frac{1}{2}$BAQ ($= \frac{1}{2}x$ suppose).

Let $AB = d$, then we have

$$BP = PQ = d \sin \tfrac{1}{2}x, \quad \text{and} \quad QA = d \cos x;$$

therefore area of quadrilateral AQPB

$$= \tfrac{1}{2}BP.PQ.\sin BPQ + \tfrac{1}{2}AB.AQ.\sin BAQ$$
$$= \tfrac{1}{2}d^2(\sin^2 \tfrac{1}{2}x \sin x + \cos x \sin x)$$
$$= \tfrac{1}{8}d^2(2\sin x + \sin 2x),$$

which, since d is constant, is by simple differentiation, found to be a maximum when x is 60°; therefore QBA = 30°, and consequently ABP = 60°; hence the quadrilateral is half a regular hexagon.

Indeterminates as a means of determining Possibility.

By G. O. Hanlon.

If we heat a copper bar, and lay it with one end on a cold piece of lead, a musical sound will be produced. It is not easy to find a satisfactory explanation of this phenomenon; but it is probably due to the fact, that the waves of heat from the copper pass through the lead in an irregular manner, owing to the want of equal density through either metal. A repulsive power seems to exist between two substances of different temperature, so that the copper would be at first lifted slightly above the lead. Now the repulsive power of the two metals, which varies as the difference of their temperatures, would, if the metals were respectively homogeneous, uniformly decrease, and the result would be that the closeness of contact would uniformly increase. But the decrease of difference of temperature

not being uniform, owing to the difference of density, the distance between the copper and the lead would not become uniformly less, but would vary with each check the heat received in passing from one to the other. This might possibly produce a knocking together of the metals, and the consequent vibrations would account for the sound. Now, there is little doubt but that no two bodies in nature possess exactly the same temperature. Every metal has a different specific heat, so that a difference in the temperature of substances must be produced even by the passage of the sun across their hemisphere. And when we consider that the motion, which is shown to be produced by difference of temperature, can never altogether cease, if we allow any coefficient of elasticity, however small, to any two metals in contact, we are irresistibly forced to the conclusion that some slight vibration is produced when any substance lies on another. And may we not fancy the hearing of some animalculæ to be so acute as to enable them to detect a delightful harmony all around, coming from stones and metals apparently in a state of rest? Insects without sight might be guided to their habitations or to their food by their different musical notes. Could Shakespeare have been thinking of this when he says?—

> "There's not the smallest orb which thou behold'st
> But in his motion like an angel sings,
> Still quiring to the young-eyed cherubins.
> Such harmony is in immortal souls;
> But, whilst this muddy vesture of decay
> Doth grossly close it in, we cannot hear it."

Now, whatever is the real nature of a statical pressure due to gravity, whether this constant motion exists or not, it is plain that the phenomenon of weight may be represented by a quantity of matter moving with an infinitely small velocity. The velocity of a body apparently at rest is then the value v would take in the expression $v = gt$, when t is infinitely small. Let it, in this case, take the value w; we propose, in this paper, to call such a quantity an *indeterminate*. We might, without interfering with the reasoning, write 0 instead of w; but it is not the custom to treat 0 as a quantity, and multiply and divide by it as we shall do here; and it is quite possible that, although such a symbol as w may be by itself useless for calculating purposes, its *ratio* to another indeterminate might be finite, as in the case of $0 : 0$. Our theory will then be, that when k and k' are constants, $\frac{kw}{k'}$ is indeterminate, but that $\frac{kw}{k'w'}$ *may* be determinate. This will explain the difficulty of comparing the effect of a moving body with the pressure from a weight, since the motion of one is given and finite, and the motion of the other infinitely small. This difficulty lies at the root of all such comparisons; but the very difficulty points out the instances in which a comparison can be effected. We would here parenthetically remark, that we know as little of weight *per se* as we do of momentum *per se*. It has been asserted that we know what weight is, but that we do not know what momentum is, and that therein lies the difficulty of comparing the two. But we contend that there is as little real conception of the nature of a pound weight as there is of a pound weight moving with a given velocity; in fact, that all we know of the weight of a body is the number of units of what we agree to call a pound, which it contains.

Now let us, on the supposition that the phenomenon of weight can be represented by matter moving with an infinitely small velocity, consider the expression $\frac{mv}{m'v'}$, where m and m' are masses, and v and v' their respec-

tive velocities. It is clear that if we suppose the four quantities to vary separately as the time, the expression, when the time is nothing, can take (recollecting our convention about w) any of the forms

$$\frac{wv}{m'v'},\ \frac{mw}{m'v'},\ \frac{mv}{wv'},\ \frac{mv}{m'w}.$$

All these values are indeterminate, since w only enters the numerator or denominator. The first and third, and second and fourth, are reciprocal in form; and the two last-mentioned may be represented by $\frac{mw}{m'v'}$. Now, the numerator mw represents a weight, since $v = gt$, and v becomes w when $t = 0$; while the denominator $m'v'$ represents a definite body moving with a definite velocity. This would seem to show that we cannot compare the pressure of a weight with the momentum of a moving body.

But the expression $\frac{mv}{m'v'}$, if we suppose the quantities to vary in couples as the time, may also take the forms

$$\frac{ww'}{m'v'},\ \frac{mv}{ww'},\ \frac{mw}{m'w'},\ \frac{mw}{w'v'},\ \frac{wv}{m'w'},\ \frac{wv}{w'v'}.$$

The first two of these are indeterminate, while the fourth and fifth are reciprocal in form. The ratios, therefore, which appear soluble reduce to three, namely,

$$\frac{mw}{m'w'},\ \frac{mw}{w'v'},\ \frac{wv}{w'v'}.$$

The first of these is merely the ratio between two given weights, and is, we know, calculable. We have, therefore, only the two last soluble forms to consider. To do this, we are led to ask, Is there any other kind of motion which partakes of the nature of an indeterminate. The force of a fluid in motion bears a very strange relation to that of a body at rest, inasmuch as the force of the latter is dependent on the quantity of matter in the body, which is definite, and the velocity, which is an indeterminate; while the force in the former is dependent on the velocity, which is definite, and the quantity of matter, which is an indeterminate. Human thought cannot determine the quantity of matter which, in an infinitely short time, strikes the hand placed in a running stream. The quantity varies as the time, and should become nothing when the time is nothing. In fact, it is w, an indeterminate. We have now an instance of wv (that is, an indefinitely small amount of matter moving with a definite velocity); it is to be found in the motion of a fluid. Therefore the apparent solubility of the expressions

$$\frac{mw}{w'v'} \text{ and } \frac{wv}{w'v'}$$

leads us to assert that we may be able to compare the force in a flowing stream ($w'v'$) with a statical weight (mw), or to compare the forces in two streams (wv and $w'v'$) of the same or different densities and moving with different velocities. We shall now show that both these comparisons are possible. If we put a hole in the end of a tank of water placed on wheels, and allow the water to flow freely out of the orifice, the tank will move in a direction opposite to the motion of the stream. But if we *attach a vessel to the tank* to receive the flowing liquid, then no motion of the tank will take place, since the action and reaction must be equal. Now, if we consider the force on the tank which would produce motion if the vessel was

not attached to it, we shall find it to be the weight of a column of water equal in base to the orifice and in height to the height of water above the centre of the orifice; while the force which acts against the side of the receiving vessel (by hypothesis equal to this weight) is a stream of water with area of base equal to half that of the orifice, and moving with a velocity due to the height of water. We can unquestionably assert that this moving stream and the statical weight are equal, and all the quantities can be rigorously calculated. Indeed, we could, by these considerations, determine what weight would remain exactly balanced at a given height above the ejection-pipe of a fountain, the velocity of ejection and area of orifice being given.

That we can compare the forces in two flowing streams, is easy of proof in many ways; but it follows at once from the above, since we can find the equivalent statical weight for each stream.

We have not touched on the meaning of the expression ww', and the length this paper has reached will not allow of it; therefore, in tabulating the results we have arrived at, we shall only take those terms which do not contain that expression; nor shall we touch on the results obtainable by supposing the four quantities m, v, m' and v' to vary in triplets as the time. Our results are

$$\frac{wv}{m'v'} = \frac{\text{force in a stream}}{\text{momentum of a mass}}, \text{ is indeterminate;}$$

$$\frac{mw}{m'v'} = \frac{\text{statical weight}}{\text{momentum of a mass}}, \text{ is indeterminate;}$$

$$\frac{mw}{m'w'} = \frac{\text{statical weight}}{\text{statical weight}}, \quad \text{is determinate;}$$

$$\frac{mw}{w'v'} = \frac{\text{statical weight}}{\text{force in a stream}}, \quad \text{is determinate;}$$

$$\frac{wv}{w'v'} = \frac{\text{force in a stream}}{\text{force in a stream}}, \quad \text{is determinate.}$$

We do not offer any opinion as to the correctness of the views put forth here. It may be that the theory is only accidentally correct in the foregoing instances; but it is no doubt remarkable that the method should point out cases which are soluble, while it declares impossible what has certainly never been accomplished.

3112. (Proposed by the Rev. J. Blissard.)—Prove that

$$\frac{1}{x} - \frac{1}{x+1} + \frac{1}{x+2} - \text{\&c. ad inf.}$$

$$= \frac{1}{x}\cdot\frac{1}{2} + \frac{1}{x(x+1)}\cdot\frac{1}{2^2} + \frac{1.2}{x(x+1)(x+2)}\cdot\frac{1}{2^3} + \dots + \frac{1.2\dots n}{x(x+1)\dots(x+n)} + \text{\&c.}$$

Solution by the Rev. G. H. Hopkins, M.A.

The series $\frac{1}{x} - \frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x+3} + \text{\&c.}$ to infinity may be written

$$\left(1-\frac{1}{2}\right)^{-1}\cdot\frac{1}{x}\cdot\frac{1}{2}-\left(1-\frac{1}{2}\right)^{-2}\cdot\frac{1}{x+1}\cdot\frac{1}{2^2}+\left(1-\frac{1}{2}\right)^{-3}\cdot\frac{1}{x+2}\cdot\frac{1}{2^3}-\text{\&c. to infinity}$$

$$=\frac{1}{x}\cdot\frac{1}{2}+\frac{1}{x}\cdot\frac{1}{2^2}+\frac{1}{x}\cdot\frac{1}{2^3}+\frac{1}{x}\cdot\frac{1}{2^4}+\frac{1}{x}\cdot\frac{1}{2^5}+\text{\&c. to infinity}$$

$$-\frac{1}{x+1}\cdot\frac{1}{2^2}-\frac{1}{x+1}\cdot\frac{2}{2^3}-\frac{1}{x+1}\cdot\frac{3}{2^4}-\frac{1}{x+1}\cdot\frac{4}{2^5}-\text{\&c. to infinity}$$

$$+\frac{1}{x+2}\cdot\frac{1}{2^3}+\frac{1}{x+2}\cdot\frac{3}{2^4}+\frac{1}{x+2}\cdot\frac{6}{2^5}+\text{\&c. to infinity}$$

$$-\frac{1}{x+3}\cdot\frac{1}{2^4}-\frac{1}{x+3}\cdot\frac{4}{2^5}-\text{\&c. to infinity}$$

$$+\frac{1}{x+4}\cdot\frac{1}{2^5}+\text{\&c. to infinity}$$

$$\text{\&c.}\qquad\text{\&c.}\qquad\text{to infinity}$$

$$=\frac{1}{x}\cdot\frac{1}{2}+\frac{1}{2^2}\left(\frac{1}{x}-\frac{1}{x+1}\right)+\frac{1}{2^3}\left(\frac{1}{x}-\frac{2}{x+1}+\frac{1}{x+2}\right)$$

$$+\frac{1}{2^4}\left(\frac{1}{x}-\frac{3}{x+1}+\frac{3}{x+2}-\frac{1}{x+3}\right)+\text{\&c.}$$

$$=\frac{1}{x}\cdot\frac{1}{2}+\frac{1}{x(x+1)}\cdot\frac{1}{2^2}+\frac{1.2}{x(x+1)(x+2)}\cdot\frac{1}{2^3}+\frac{1.2.3}{x(x+1)(x+2)(x+3)}\cdot\frac{1}{2^4}+\text{\&c.}$$

3110. (Proposed by M. W. Crofton, F.R.S.)—Considering $x^{-1}D$ as a simple symbol, prove that

$$e^{hx^{-1}D}f(x)=f\left\{(x^2+2h)^{\frac{1}{2}}\right\},\quad\text{also that}\quad \phi(x^{-1}D)f(x)=\Big]_x^{x^2}\phi(2D)f(x^{\frac{1}{2}}),$$

the symbol $\Big]_x^{x^2}$ indicating the substitution of x^2 for x in what follows, as introduced by Sarrus.

Solution by Samuel Roberts, M.A.; J. J. Walker, M.A.; *and others.*

Put θ for x^2, so that $x^{-1}D$ is equivalent to $2\dfrac{d}{d\theta}$. Then

$$\phi(x^{-1}D)fx=\phi\left(2\frac{d}{d\theta}\right)f(\theta^{\frac{1}{2}});$$

and after operating we have to put x^2 for θ. This is plainly equivalent to the latter form of the question. The first form is a particular case obtained

by means of $\quad e^{2h\frac{d}{d\theta}}f(\theta^{\frac{1}{2}})=f(\theta+2h)^{\frac{1}{2}}=f(x^2+2h)^{\frac{1}{2}}.$

2905. (Proposed by the Rev. T. P. Kirkman, M.A., F.R.S.)—

Mr. Punch's renown
In London town
Brought up in dozens
His country cousins;
Twenty-eight ladies, pretty and shy,
Twenty-one gentlemen, six feet high.

Quoth he, "I invite
Four couples a-night,
A belle with a beau,
Whenever you choose;
If only, you know,
Just now, in the session,
You have the discretion
This rule to use,—
That never a pair
Of you all shall share
Together twice my evening fare."

Then smil'd and bow'd
The happy crowd,
In full content;
And beau with belle,
The hungry sinners,
In eights they went,
And polished off well
Just twenty-one dinners.

They were loth to leave when all was o'er,
And the rule forbade an octave more.
Then went Mr. Punch on,
"I bid you to luncheon,
A beau with a belle,
In couples three;
But look to it well
I never see
Two meet, who have met at table with me."

The joy was loud
Of the happy crowd;
And twenty-eight noons,
In sixes merry,
They plied his spoons
And drank his sherry.

Then, to the fair who alone, as yet,
In his banquet-hall had never met,
He said, "My dears, (it can't be improper,) ar-
range to go with me all to the Opera;
Come only in flocks of pairs never able
To meet in my box or meet at my table."

Then for eight nights,
Oh, all in their best
So charmingly drest,
Came ravishing sights,
In bevies of seven;
And, girt and caress'd
By the dear delights
Mr. Punch was blest
With peeps at heaven.

Have you the skill
The lists to fill,
And of forty-nine
All pairs combine?

Solution by the Proposer.

The problem is a simple case of the general theorem, "When N is any prime number, N^2 elements can be thrown into $\frac{1}{2}N(N+1)$ $(N-1)$-plets, $N+1$ N-plets, and $\frac{1}{2}N(N-1)$ $(N+1)$-plets, so as to exhaust once and once only the duads with the N^2 elements," which is given for the first time, I believe, on p. 214 of the *Memoirs of the Literary and Philosophical Society of Manchester*, Vol. II., Third Series, 1865.

The required arrangements of the 21 gentlemen,

ABCDEFGHIJKLMNPQRSTUV,

with the 28 ladies, $abcdefghijklmnpqrstuvwxyz\alpha\beta\gamma$,

are thus taken from the same page:—

$$
\begin{array}{llll}
a\mathrm{Q}e\mathrm{N}b\mathrm{R}f\mathrm{S} & t\mathrm{E}j\mathrm{T}f\mathrm{I}\beta\mathrm{B} & x\mathrm{C}s\mathrm{E}h\mathrm{U}e\mathrm{F} & a\mathrm{A}y\mathrm{G}i\mathrm{R}h\mathrm{I} \\
a\mathrm{L}g\mathrm{V}c\mathrm{T}h\mathrm{J} & t\mathrm{V}k\mathrm{C}n\mathrm{R}w\mathrm{K} & x\mathrm{A}z\mathrm{P}q\mathrm{S}n\mathrm{T} & a\mathrm{P}p\mathrm{L}k\mathrm{N}s\mathrm{B} \\
a\mathrm{K}i\mathrm{P}d\mathrm{U}j\mathrm{M} & t\mathrm{D}p\mathrm{F}y\mathrm{J}d\mathrm{S} & x\mathrm{M}k\mathrm{J}\beta\mathrm{G}b\mathrm{H} & a\mathrm{E}c\mathrm{H}u\mathrm{K}m\mathrm{S}
\end{array}
$$

$\gamma BuRlJqU$	$vDfCgGuP$	$rEqGwLdQ$
$\gamma FgAjHwN$	$vA\beta UpQmV$	$rBzHeViD$
$\gamma DcIsMnQ$	$vFlIzKbL$	$rTlCmNyM$

$bCcAdB$	$gQxByK$	$xVdNuL$	$tPhHlQ$
$aDlAkE$	$\gamma C\beta SiL$	$eImJwP$	$iFkQuT$
$aGmBnF$	$wTbUaD$	$\gamma TeGpK$	$lGjVsS$
$aHqCpI$	$\gamma PbVyE$	$fHnLyU$	$jDmLxR$
$fJsArK$	$rF\beta PcR$	$qFfVaM$	$gIkSrU$
$eLuAtM$	$cNtGzU$	$gEpMzR$	$vJnEiN$
$hMwBvS$	$vTdRsH$	$hDqK\beta N$	$jJaCzQ$

$a\gamma txvr\alpha$	$\gamma dzhkmf$	$xwcfpli$	$rbujnpk$
$au\beta ywzs$	$tbsimqg$	$vcyeqkj$	$\alpha\beta ledgn$

These arrangements are all deduced from the first octuplet $aQeNbRfS$, by a rigorous tactical method in the theory of groups, which it is impossible to give here. I know of no key, obtainable from ordinary algebra or combinations, to these multiplets; but I have no doubt that one may be constructed, of a formidable size, by the theory of congruences.

[The first problem of this kind that seems to have attracted much attention, if not indeed the first that was ever published, is the famous "puzzle of the fifteen young ladies," which was proposed in prose by Mr. Kirkman in the *Ladies' and Gentlemen's Diary* for 1850, and has since been thus versified by a lady:—

A governess of great renown
Young ladies had fifteen,
Who promenaded near the town,
Along the meadows green.

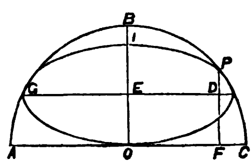
But as they walked
They tattled and talked,
In chosen ranks of three,
So fast and so loud,
That the governess vowed
It should no longer be.

So she changed them about,
For a week throughout,
In threes, in such a way
That never a pair
Should take the air
Abreast on a second day;
And how did the governess manage it, pray?

The first properly mathematical solution that appeared was given by Mr. Kirkman, in the *Philosophical Magazine* for March, 1862.]

3092. (Proposed by A. Martin.)—Find the average area of all the ellipses that can be inscribed symmetrically in a given semi-ellipse.

Solution by the Proposer.



Let ABCO be the given semi-ellipse, and GIPO any ellipse inscribed symmetrically in it. Put A and B for the semi-axes of the given ellipse, α and β for the semi-axes of the inscribed ellipse, and x and y for the coordinates of the point P, the origin being at O, the centre of the given ellipse. Then the equations to the given and inscribed

ellipses are, respectively,

$$A^2y^2 + B^2x^2 = A^2B^2, \quad \alpha^2(y-\beta)^2 + \beta^2x^2 = \alpha^2\beta^2 \ \ldots\ldots\ldots\ldots\ldots (1, 2).$$

From (1), $\dfrac{dy}{dx} = -\dfrac{B^2x}{A^2y}$; from (2), $\dfrac{dy}{dx} = -\dfrac{\beta^2x}{\alpha^2(y-\beta)}$.

Since the ellipses have a common tangent at P, these values of $\dfrac{dy}{dx}$ must be equal; therefore $\dfrac{B^2}{A^2y} = \dfrac{\beta^2}{\alpha^2(y-\beta)}$, whence $y = \dfrac{B^2\alpha^2\beta}{B^2\alpha^2 - A^2\beta^2}$.

Subtracting B^2 times (2) from β^2 times (1), and substituting the above value of y, we find, after reduction, $\beta^2 = \dfrac{B^2\alpha^2(A^2-\alpha^2)}{A^4}$ ....(3).

The area of the inscribed ellipse is $\pi\alpha\beta$, and the average area required is $\dfrac{2}{B}\displaystyle\int_0^{\frac{1}{2}B} \pi\alpha\beta\, d\beta$, supposing the centres of the inscribed ellipses to be uniformly distributed along the semi-minor axis OB.

But, from (4), $\quad \beta\, d\beta = \dfrac{B^2\alpha(A^2 - 2\alpha^2)\, d\alpha}{A^4}$,

and the limits of α [obtained from (3) by putting $\beta = 0$ and $\beta = \frac{1}{2}B$] are A and $\frac{1}{2}A\sqrt{2}$. Hence the average area required is

$$\frac{2\pi B}{A^4}\int_A^{\frac{1}{2}A\sqrt{2}} \alpha^2(A^2 - 2\alpha^2)\, d\alpha = \frac{\pi AB}{15}(2+\sqrt{2}).$$

[When A = B, we have one of the cases of the Editor's Question 2577; see *Reprint*, Vol. X., p. 84.]

NOTE ON QUESTION 2823. *By* MORGAN JENKINS, M.A.

This question, proposed by Professor Sylvester, is as follows:—

"Show that, on a chess-board, the chance of a rook moving from one square to another without changing colour is $\frac{3}{7}$; but that, without altering the equality of the number of the black and white squares, but only the manner of their distribution, the chance may be made equal to $\frac{1}{2}$."

For the latter part of the question, if

$x_1, x_2, x_3, \ldots x_8$ be the number of black squares in the respective rows,
$y_1, y_2, \ldots\ldots y_8$ " " " columns,

the *equations* which it is necessary to solve are

$x_1 + x_2 + \ldots\ldots + x_8 = y_1 + y_2 + \ldots\ldots y_8 = 32$ arising from the total number of black squares being given .... (α),

and

$x_1(x_1-1) + \ldots + (8-x_1)(7-x_1) + \ldots + y_1(y_1-1) + \ldots + (8-y_1)(7-y_1) + \ldots$
$= \frac{1}{2}\left\{(8\times 7)\,8 + (8\times 7)\,8\right\} = 64\times 7$ arising from the value of the probability of the specified event being given.

The last equation reduces to the simpler and still symmetrical form

$$x_1(8-x_1)+\ \dots\ +y_1(8-y_1)+\ \dots\ =32\times 7 \quad \dots\dots\dots\dots\dots \quad (\beta).$$

A system of inequalities, found more generally in the solution of Question 3119, must also be satisfied in order that the arrangement represented by two sets of x's and y's may be possible.

I have tabulated below the solutions (I think complete) of the preceding equations. They include the solutions given by Mr. Collins and myself in Vol. XI. (pp. 50—52) of the *Reprint*. I find, moreover, by application of the necessary conditions of inequality, that the arrangements represented by every one of these 364 solutions are possible. This suggests the question, more generally for a rectangle divided regularly into n rows and m columns, and containing p black squares, whether there is any relation by which the satisfaction of the additional equation necessary to secure a given probability of moving a rook without changing colour, will at the same time secure the satisfaction of the conditions of inequality necessary for possibility of arrangement.

Proceeding with the question before us, let

$a_0, a_1, \dots a_8$ be the numbers of rows containing respectively $0, 1, \dots 8$ black squares each,

$b_0, b_1, \dots b_8$ the numbers of like columns;

let $\quad a_0+b_0=c_0,\ \ a_1+b_1=c_1,\ \ \dots\dots\ \ a_8+b_8=c_8,$

$$c_0+c_8=d_0,\ \ c_1+c_7=d_1,\ \ c_2+c_6=d_2,\ \ c_3+c_5=d_3.$$

Then we have

$$a_0+a_1+\dots+a_8=8 \ \dots\dots\ (1), \quad \text{and} \quad b_0+b_1+\dots+b_8=8 \ \dots\dots\ (2),$$

and from (α),

$$a_1+2a_2+\dots+8a_8=32 \ \dots\dots\ (3), \quad \text{and} \quad b_1+2b_2+\dots+8b_8=32 \ \dots\dots\ (4).$$

Instead of (3) and (4), we have from (1) and (2) the symmetrical equations

$$4a_0+3a_1+2a_2+a_3=4a_8+3a_7+2a_6+a_5 \ \dots\dots\dots\dots\dots\dots\dots\ (5),$$

and

$$4b_0+3b_1+2b_2+b_3=4b_8+3b_7+2b_6+b_5 \ \dots\dots\dots\dots\dots\dots\dots\ (6),$$

and hence, by addition,

$$c_0+c_1+\dots+c_8 \ \text{ or } \ d_0+d_1+d_2+d_3+c_4=16 \ \dots\dots\dots\dots\dots\ (7),$$

and

$$4c_0+3c_1+2c_2+c_3=4c_8+3c_7+2c_6+c_5 \ \dots\dots\dots\dots\dots\dots\ (8),$$

also from (β),

$$7d_1+12d_2+15d_3+16c_4=32\times 7 \ \dots\dots\dots\dots\dots\dots\dots\dots\ (9).$$

Completely solving (9) in positive integers, picking out those whose sum does not exceed 16, and filling up the value of d_0 from (7), 19 solutions are found.

Again, in these 19 solutions form the various values of $4d_0+3d_1+2d_2+d_3$; they are found to range from 8 to 20 inclusive. Take, by way of example, one of the 19 solutions in which d_0, d_1, d_2, d_3, c_4 and $4d_0+3d_1+2d_2+d_3$ $=1, 0, 2, 8, 5,$ and 16 respectively.

Form the various solutions of

$$4c_0+3c_1+2c_2+c_3=8, \quad i.e. \text{ half } 4d_0+3d_1+2d_2+d_3,$$

and pick out those whose respective sums are 1, 0, 2, 8; for example,

$$(1, 0, 0, 4)+(0, 0, 2, 4)=(1, 0, 2, 8);$$

then, by virtue of (8), we may take

$$(1, 0, 0, 4, 5, 4, 2, 0, 0), \quad \text{or its reverse} \quad (0, 0, 2, 4, 5, 4, 0, 0, 1),$$

for the successive values of c_0, c_1, ... to c_3 respectively; c_4 being filled up from (7). In this way we obtain 5 sets of values of the c's which are unchanged by reversion, and 36 which are changed; that is, $5 + (2 \times 36) = 77$ sets of values of the c's.

Lastly, forming all the solutions of $4a_0 + 3a_1 + 2a_2 + a_3 = k$, and taking any solution with itself or some other solution written in reverse order, and inserting in the middle the proper value of a_4 from (1), we then try whether this set will subtract from a set of the c's: if it will, the remainders are the corresponding set of b's. Thus, taking $k = 5$, we have from the solutions 1, 0, 0, 1 and 0, 0, 2, 1 the following set of a's: (1, 0, 0, 1, 3, 1, 2, 0, 0); and looking amongst the sets of c's, this will subtract from the above-written (1, 0, 0, 4, 5, 4, 2, 0, 0), leaving for the set of b's (0, 0, 0, 3, 2, 3, 0, 0, 0).

It is not necessary to take values of k greater than 5, because the greatest value of $4c_0 + 3c_1 + 2c_2 + c_3$ is $\frac{20}{2}$ or 10; and therefore, if $k > 5$, then $4b_0 + 3b_1 + 2b_2 + b_3$ would be < 5, say l; and by interchanging the words row and column, as might be done from symmetry, we should interchange k and l.

The following are the associated pairs of sets of a's and b's, in which either one or both of the sets may be reversed. Of these pairs, in 16 both sets are reversible without change, in 76 one set only is changed, and in 49 both sets are changed by reversion. Thus the total number of numerical solutions is $16 + (76 \times 2) + (49 \times 4) = 364$.

The solution given by Mr. Collins is indicated by the associated pair (0, 0, 0, 0, 8, 0, 0, 0, 0), (1, 0, 0, 0, 6, 0, 0, 0, 1). In the table, the sets which are not changed by reversion are marked thus †.

†(000080000)	(100060001)†
	101030300
	101023010
	100221110
	020031110
	101103200
	100301300
	020111300
	020104010
	012012110
	004000400†
†(000161000)	(000220101)
	003022001
	010312001
	011041001
	002301020
	003030020
	010320020
	003110210
	011202110†
	012004010
†(000242000)	(001410101)
	002212001
	010231001
	002220020
	002300210
	003102110
	011121110†

†(001060100)	(001321001)
	002050001
	000600020
	002211110
	003013010
	010303010†
	011032010
	003020300†
(000250100)	(000600101)
	001402001
	002131001
	001410020
	003021110
	011113010
	003101300†
	010150001
	011040110†
†(000323000)	(000600101)
	001402001
	002131001
	010150001
	001410020
	003021110
	011040110†
	011113010
	003101300†

†(001141100) (000511001)
001240001
001401110
002130110
002203010
010222010†
002210300

†(010060010) (000421010)
001150010
001303100†
002032100
002105000

(000331100) (001321001)
002050001
000600020
002211110
003013010
011032010
003020300†
010303010†

(001150010) (000510110)
001312010
002041010
010060010†
002121210

(000340010) (000430001)
001320110
002122010
010141010†
001400300
002202200†
003004100
011023110

†(000404000) (001321001)
002050001
000600020
003013010
011032010
002211110
010303010†

†(001222100) (000430001)
001320110
002122010
010141010†
001400300
002202200†

†(002040200) (001231010)
010205000
002113100
002040200†

†(010141010) (002024000)

(000412100) (100042100)
100115000
011031200
010222010†
011104100
010302200
003012200

(000420200) (100054000)
000430001
011023100
002122010
003004100
002202200†

(000421010) (010140200)
011015000
010213100
002121200†

(001231010) (001231010)
002040200
010205000
002113100

(001230200) (010140200)
011015000
010213100
002121200†

(002041010) (002105000)
001303100

†(001303100) (000510110)
001312010
002121200†

†(002121200) (000501200)
001303100

(000501200) (000510110)
001312010

(001321200) (001301200)

3111. (Proposed by C. W. MERRIFIELD, F.R.S.)—In the rectangular hyperbolic paraboloid, using orthogonal projections by lines parallel to the principal parabolic axis, prove that (1) the areas of any two portions of surface, which have similar and equal projections at equal distances from the axis, are equal to one another; and (2) if the projection be any figure symmetrical to the projections of any two right-line generators, the corresponding cylindrical volume shall be the product of the projected area into the ordinate at its middle point.

Solution by the PROPOSER.

1. Let the equation of the surface be $kz = xy$. Then the area corresponding to any projected area whatever $\iint d\alpha\, d\beta$ will be $\iint d\alpha\, d\beta \sec\tau$,

where $$\sec\tau = \left(1 + \frac{dz^2}{dx^2} + \frac{dz^2}{dy^2}\right) = \left(1 + \frac{x^2+y^2}{k^2}\right)^{\frac{1}{2}} = \left(1 + \frac{r^2}{k^2}\right)^{\frac{1}{2}},$$

which is independent of θ, and may therefore be made to revolve round the axis of z without change.

Corollary.—Any annulus of the surface about the axis of z may be measured by a parabolic arc.

2. $4xy \equiv \Sigma(x \pm h)(y \pm k)$. The proposition follows at once from the summation of this identity within the limits stated.

3054. (Proposed by the Rev. W. ROBERTS, M.A.) — The cone $x^2\cot^2\alpha + y^2\cot^2\beta - z^2 = 0$ intersects the sphere $x^2+y^2+z^2-a^2 = 0$ in a sphero-conic. Show that the equation of the tubular surface, which is the envelope of a sphere of constant radius k, whose centre moves on this sphero-conic, is had by equating to zero the discriminant of the following cubic in λ, $$\frac{4a^2\sin^2\alpha\, x^2}{P^2 + 4\lambda a^2\cos^2\alpha} + \frac{4a^2\sin^2\beta\, y^2}{P^2 + 4\lambda a^2\cos^2\beta} - \frac{z^2}{\lambda} = 1,$$
where $P = x^2+y^2+z^2+a^2-k^2$.

Solution by J. J. WALKER, M.A.

The problem is, to find the envelope of
$$(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 - k^2 = 0 \quad\text{.....................} \quad (1),$$
ξ, η, ζ being subject to the conditions
$$\xi^2\cot^2\alpha + \eta^2\cot^2\beta - \zeta^2 = 0 \;......\; (2), \quad\text{and}\quad \xi^2+\eta^2+\zeta^2-a^2 = 0 \;......\; (3).$$
From (1) and (3) it readily follows that $2z\zeta = P - 2(x\xi + y\eta)$ .... (4),
and from (2) and (3) that $\xi^2\operatorname{cosec}^2\alpha + \eta^2\operatorname{cosec}^2\beta - a^2 = 0$ .... (5).
Again, from (3) and (4),
$$4z^2(a^2-\xi^2-\eta^2) = 4z^2\zeta^2 = P^2 - 4P(x\xi+y\eta) + 4(x\xi+y\eta)^2 \;.......\; (6).$$

Introducing ω for 1, to make them homogeneous in $\xi\eta\omega$, (6) and (5) become

$$u = 4(x^2+z^2)\xi^2 + 4(y^2+z^2)\eta^2 + (P^2-4a^2z^2)\omega^2 - 4Py\eta\omega - 4Px\omega\xi + 8xy\xi\eta = 0 \quad\ldots\ldots\ldots\ldots\ldots (7),$$

and $$v = \operatorname{cosec}^2\alpha\,\xi^2 + \operatorname{cosec}^2\beta\,\eta^2 - a^2\omega^2 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

respectively.

The required envelope may now be found (in a manner suggested by Mr. Burnside, *Reprint*, Vol. XI., p. 42) by equating with zero the discriminant with respect to λ' of the discriminant of $u+4\lambda'v$ with respect to $\xi\eta\omega$. This latter discriminant, equated with zero, gives

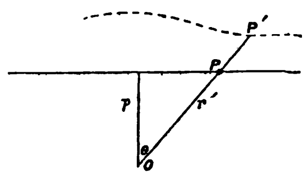
$$\frac{4a^2(z^2+\lambda')\,x^2}{P^2(z^2+\lambda'\operatorname{cosec}^2\alpha)} + \frac{4a^2(z^2+\lambda')\,y^2}{P^2(z^2+\lambda'\operatorname{cosec}^2\beta)} + \frac{4a^2(z^2+\lambda')}{P^2} = 1 \quad\ldots\ldots (9).$$

Assuming $\lambda' = -\dfrac{P^2z^2\lambda^{-1}}{4a^2} - z^2$, this cubic is transformed into that given in the question; and its discriminant is not altered (to a factor) by this transformation.

3073. (Proposed by the Rev. E. HILL, M.A.)—Heat is radiating in all upward directions from a focus below a plain on the earth's surface. At the end of a certain time the heat at any point varies as the inverse square of the distance from the focus. Show (1) that the surface has now assumed the form $p\sec\theta = r - k\tan^{-1}\dfrac{r}{k}$, p and k being constants, and (2) that this form is convex.

Solution by H.

1. For the expansion of plain. The heat at a distance r being $= \dfrac{\mu}{r^2}$, an element $\delta r'$ at this point becomes



$$\left(1+\frac{a\mu}{r^2}\right)\delta r';$$

therefore $$\delta r = \left(1+\frac{a\mu}{r^2}\right)\delta r',$$

$$\delta r' = \frac{1}{1+\dfrac{a\mu}{r^2}}\,\delta r = \frac{r^2}{k^2+r^2}\,\delta r = \left(1-\frac{k^2}{k^2+r^2}\right)\delta r.$$

Integrating, the limits of r are 0 and r; those of r', the natural radius, 0 and $p\sec\theta$; therefore $p\sec\theta = r - k\tan^{-1}\dfrac{r}{k}$.

2. To prove convexity. $p + k\tan^{-1}\dfrac{r}{k}\cdot\cos\theta = r\cos\theta$.

Put $r\cos\theta = x$; $\therefore\ p + k\tan^{-1}\left(\frac{x}{k\cos\theta}\right).\cos\theta = x$;

$$\therefore\ \frac{dx}{d\theta} = -k\tan^{-1}\left(\frac{x}{k\cos\theta}\right)\sin\theta + \frac{k^2\cos\theta}{x^2+k^2\cos^2\theta}\left(\frac{dx}{d\theta}\cos\theta + x\sin\theta\right);$$

$$\therefore\ \frac{dx}{d\theta}\left\{1 - \frac{k^2\cos^2\theta}{x^2+k^2\cos^2\theta}\right\} = k\sin\theta\left\{-\tan^{-1}\left(\frac{x}{k\cos\theta}\right) + \frac{kx\cos\theta}{x^2+k^2\cos^2\theta}\right\}.$$

The coefficient of $\frac{dx}{d\theta}$ is positive. $k\sin\theta$ is positive. Since x is considerable and k very small (involving the coefficient of expansion), therefore $\tan^{-1}\left(\frac{x}{k\cos\theta}\right)$ is about $\frac{\pi}{2}$, and $\frac{k\cos\theta}{x^2+k^2\cos^2\theta}$ is very small; therefore the factor on the right is negative; therefore $\frac{dx}{d\theta}$ is negative; therefore the curve is convex.

2973. (Proposed by J. J. Walker, M.A.)—The equation to the lines joining the centre of an ellipse $(b^2x^2 + a^2y^2 = a^2b^2)$ with three points, the osculating circles at which cointersect at (x', y'), a fourth point on the ellipse, is
$$y'x\,(b^2x^2 - 3a^2y^2) = x'y\,(a^2y^2 - 3b^2x^2).$$
Prove this, and show that when (x', y') is one of the four points common to the ellipse and either of a certain pair of concentric circles, the three lines above, together with that joining the centre with (x', y'), form a harmonic pencil.

Solution by the Rev. J. Wolstenholme, M.A.; R. W. Genese; *and others.*

If θ be the eccentric angle of one of the points (x, y), that of (x', y') is -3θ. The required equation is then found by eliminating θ from the equations
$$\frac{y}{x} = \frac{b}{a}\tan\theta,\quad \frac{y'}{x'} = -\frac{b}{a}\tan 3\theta,$$
and is
$$\frac{y'}{x'} = \frac{b}{a}\cdot\frac{\dfrac{a^3y^3}{b^3x^3} - \dfrac{3ay}{bx}}{1 - \dfrac{3a^2y^2}{b^2x^2}} = \frac{(a^2y^2 - 3b^2x^2)\,y}{(b^2x^2 - 3a^2y^2)\,x},$$
or
$$xy'\,(b^2x^2 - 3a^2y^2) = yx'\,(a^2y^2 - 3b^2x^2).$$

For a harmonic pencil it is readily found that $\cos 4\theta = 0$, and therefore the distances, opposite signs being taken in the two cases, from the centre of (x', y'), and of (x, y), are
$$\left(\frac{a^2+b^2}{2} \pm \frac{a^2-b^2}{2\sqrt{2}}\right)^{\frac{1}{2}}.$$

2918. (Proposed by J. J. WALKER, M.A.)—From any point in a given line tangents are drawn to a cissoid, and the circle described through the points of contact: prove that the envelope of the radical axis of this and the generating circle is in general a conic passing through the cusp, which becomes an ellipse having its minor axis coincident with the tangent at cusp when the given line passes through a certain point in that tangent; one of the three conics with an axis similarly directed when the given line is perpendicular to the tangent at cusp; and the generating circle itself when both the above conditions are fulfilled.

Solution by the PROPOSER.

In a paper *On Tangents to the Cissoid*, communicated to the Mathematical Society, I have shown that the equation to the circle passing through the points of contact of tangents to the cissoid $x(x^2+y^2) = 2ry^2$, drawn from the point $x'y'$, is (the cusp being origin, and cuspidal tangent axis of x)

$$3x'(2r-x')(x^2+y^2)-2r(9x'^2+4y'^2)x+4r(2r-x')y'y+16r^2y'^2 = 0,$$

while the equation to the generating circle referred to the same axes is

$$x^2+y^2-2rx = 0.$$

The radical axis of these circles is

$$\left\{3x'(2x'-r)+2y'^2\right\}x-y'(2r-x')y-4ry'^2 = 0.$$

Suppose the given right line which is the locus of $x'y'$ to be $x = my+n$, then, substituting $my'+n$ for x' in the above equation, and arranging by powers of y',

$$\left\{2(3m^2+1)x+my-4r\right\}y'^2+\left\{3(4n-r)mx+(n-2r)y\right\}y'+3(2n-r)nx=0,$$

the discriminant of which relatively to y' is, after reduction,

$$3\left\{3m^2r^2-8(2n-r)n\right\}x^2+6mr(2r-7n)xy+(2r-n)^2y^2+48rn(2n-r) = 0,$$

the equation to the required envelope. Now the coefficient of xy vanishes either when $m=0$ or when $n=\frac{2}{7}r$; and in the latter case the equation reduces to $(49m^2+16)x^2+16y^2-32rx = 0$, which becomes the circle $x^2+y^2-2rx = 0$ if also $m=0$.

3006. (Proposed by J. J. WALKER, M.A.)—Prove that the equation to the circle circumscribing the triangle formed by tangents to the parabola $y^2-4ax=0$ drawn from (x', y'), and the chord of contact, is

$$a(x^2+y^2)-(y'^2+2a^2)x-y'(a-x')y+ax'(2a-x') = 0.$$

I. *Solution by* R. W. GENESE; R. TUCKER, M.A.; *and others.*

Transforming the origin to the point (x', y'), we get

$$y^2-4ax+2yy'+y'^2-4ax' = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

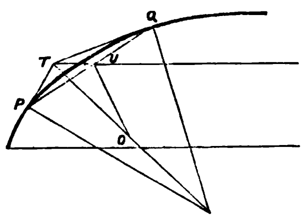
The polar of the origin is $\quad yy'-2ax+y'^2-4ax' = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$

If we can combine (1) and (2) so that the result is of form $x^2+y^2 = mx+ny$, this will be the required circle.

Subtracting (2) from (1), $y^2-2ax+yy'=0$ .... (3);
and from (2), $4a^2x^2 = (yy'+y'^2-4ax')^2$;
therefore $4a^2(x^2+y^2) = y^2(y'^2+4a^2)+2yy'(y'^2-4ax')+(y'^2-4ax')^2$
[using (3) and (2)] $= (2ax-yy')(y'^2+4a^2)+2yy'(y'^2-4ax') + (y'^2-4ax')(2ax-yy')$
$= (2ax-yy')(2y'^2+4a^2-4ax')+2yy'(y'^2-4ax')$;
therefore $a(x^2+y^2) = x(y'^2+2a^2-2ax')-yy'(a+x')$.

Returning to former origin, this takes the form given in the question.

II. *Solution by* R. Tucker, M.A.



Let T be the external point, and PQ the chord of contact; then TU, the diameter, bisects PQ, and is bisected by the curve; hence the coordinates of U are

$$\left(X = \frac{y'^2-2ax'}{2a},\quad Y = y'\right),$$

and the centre (O) of the circle lies on the perpendicular to PQ through U. Let (α, β) be the coordinates of the centre, and R the radius of the circle; then, since the equation to PQ is

$$yy' = 2a(x+x'),$$

we have $$\beta - y' = \frac{y'}{2a}\left(\alpha - \frac{y'^2-2ax'}{2a}\right) \text{ ....(i.)}$$

and $$-\frac{\beta y' - 2a(\alpha+x')}{(y'^2+4a^2)} = OU = \frac{(y'^2+4a^2)^{\frac{1}{2}}}{2a}(a+x') \text{ ....(ii.).}$$

These reduce to

$$-2a\beta y' + 4a^2\alpha = ay'^2+4a^3+x'y'^2, \quad 4a^2\beta+2a\alpha y' = 4a^2y'+y'^3-2ax'y';$$

whence $2a\beta = y'(a-x')$, and $2a\alpha = y'^2+2a^2$;

then $X^2+Y^2-2\alpha X-2\beta Y = R^2-\alpha^2-\beta^2 = x'(x'-2a)$,

since $$R^2 = \frac{(y'^2+4a^2)\left\{y'^2+(x'-a)^2\right\}}{4a^2},$$

or the equation required is the same as given in the question.

If the locus of centres is a straight line, T lies on an hyperbola; and if the locus of T be the directrix, then O lies on the parabola $\beta^2 = 2a(\alpha-a)$, and the envelope of the circles is the cubic $y^2(a+x) = x^2(2a-x)$.

[Solutions to the corresponding problem for the *ellipse* (Quest. 2949) are given on pp. 43, 44 of this volume of the *Reprint*.]

2944. (Proposed by the Rev. J. Wolstenholme, M.A.)—If O be that point in the normal to a parabola at P through which if any chord pass, it will subtend a right angle at P, PO will be bisected by the axis.

I. *Solution by* R. Tucker, M.A.; R. W. Genese; *and others.*

Referring the parabola to the tangent and normal at the point, its equation is (see Salmon's *Conics*, p. 165, ex. 2)

$$ax^2+2\lambda xy+by^2+2fy=0 \quad \text{.........................} \quad (1);$$

and the intercept on the normal (giving the fixed point in the question) is

$$=-\frac{2f}{a+b}.$$

Now the equation to the axis may be shown to be

$$(a+\lambda)\,x+(b+\lambda)\left(y+\frac{f}{a+b}\right)=0,$$

hence the truth of the theorem is obvious.

II. *Solution by* Stephen Watson; J. Dale; *and others.*

Denote P by (x', y') when referred to the principal axes of the parabola; then, for parallel axes originating at P, the equation of the parabola is

$$(y+y')^2=m\,(x-x'),$$

and this is met by the lines $y=ax$, $y=-a^{-1}x$, in two points, the equation of the line through which is

$$\frac{y+ma+2y'}{x-ma^2-2ya}=\frac{ma}{m-ma^2-2y'a} \quad \text{.........................} \quad (1),$$

also the equation of the normal at P is

$$my+2y'x=0 \quad \text{....................................} \quad (2).$$

Eliminating x from (1) and (2), there results

$$(y+2y')(ma+2y')(m-2y'a)=0,$$

therefore $y=-2y'$, which proves the theorem.

3069. (Proposed by the Editor.)—To find the sides of a triangle in rational numbers such that the three sides and the area shall be in arithmetical progression.

I. *Solution by* S. Bills; Asher B. Evans, M.A.; A. Martin; *and others.*

Let ABC be a plane triangle, and CD the perpendicular on AB. Let $CD=x$, and assume $AD=\dfrac{2pq}{p^2-q^2}x$ and $BD=\dfrac{2rs}{r^2-s^2}x$; then we shall have

$$AC=\frac{p^2+q^2}{p^2-q^2}x,\quad BC=\frac{r^2+s^2}{r^2-s^2}x,\quad AB=\left(\frac{2pq}{p^2-q^2}+\frac{2rs}{r^2-s^2}\right)x,$$

$$\text{Area ABC}=\left(\frac{pq}{p^2-q^2}+\frac{rs}{r^2-s^2}\right)x^2.$$

Let the order of the progression be AB, AC, BC, Area ABC; then, to satisfy the conditions of the question, we must have

$$AC + BC = AB + \text{Area } ABC, \quad \text{and} \quad AB + BC = 2AC;$$

or, substituting the above values, we must have

$$\left(\frac{p^2+q^2}{p^2-q^2}+\frac{r^2+s^2}{r^2-s^2}\right)x = \left(\frac{pq}{p^2-q^2}+\frac{rs}{r^2-s^2}\right)(2x+x^2) \quad \text{............ (1)},$$

$$\frac{2pq}{p^2-q^2}+\frac{2rs}{r^2-s^2}+\frac{r^2+s^2}{r^2-s^2} = \frac{2(p^2+q^2)}{p^2-q^2} \quad \text{.................. (2)}.$$

From (1) we readily find $x = \dfrac{2p^2r^2-q^2s^2}{pq(r^2-s^2)+rs(p^2-q^2)} - 2,$

and (2) becomes

$$\frac{(r+s)^2}{r^2-s^2} = \frac{2(p^2-pq+q^2)}{p^2-q^2}, \quad \text{or} \quad \frac{r+s}{r-s} = \frac{2(p^2-pq+q^2)}{p^2-q^2};$$

whence $r = \dfrac{3p^2-2pq+q^2}{p^2-2pq+3q^2}\,s$, where p, q, s may be taken at pleasure.

Take $q=0$, $s=1$; then $r=3$ and $x=4$, and we find

$$AC=4, \quad BC=5, \quad AB=3, \quad \text{Area}=6.$$

If $p=2$, $q=1$, $s=1$, then $r=3$ and $x=\frac{4}{3}$; and we find

$$AC=\tfrac{4}{3}, \quad BC=1, \quad AB=\tfrac{5}{3}, \quad \text{Area}=\tfrac{2}{3}.$$

Note.—It must be seen that the triangles found above are all right-angled; but if we take $p=3$, $q=1$, and $s=3$, then $r=11$ and $x=\frac{8}{5}$; and we find $AB=\frac{15}{7}$, $AC=2$, $BC=\frac{13}{7}$, Area $=\frac{12}{7}$; and in this example the triangle is not right-angled.

II. *Solution by the* Rev. J. Wolstenholme, M.A.

If $a-b$, a, $a+b$ be the three sides, the area must be $a+2b$, whence

$$a+2b = \left(\frac{3a}{2}\cdot\frac{a+2b}{2}\cdot\frac{a}{2}\cdot\frac{a-2b}{2}\right)^{\frac{1}{2}}, \quad \text{or} \quad 16(a+2b) = 3a^2(a-2b),$$

or

$$2b = \frac{a(3a^2-16)}{16+3a^2},$$

and giving a any value, we get a corresponding value of b. And since $16+3a^2$ is always numerically greater than $3a^2 \backsim 16$, we shall always have $a > 2b$, so that the triangle will always be possible. Thus, putting $a=4$, we get the triangle whose sides are 3, 4, 5, and area 6. This is, I think, the only solution in whole numbers.

2961. (Proposed by A. W. Ruecker.)—If an ellipse inscribed in a triangle, such that a line through the vertex is a directrix, touch the base AB in O; and conics, also touching AB in O, be inscribed in triangles formed by joining to the vertex any points P, Q in the base, such that

OP . OQ = OA . OB: then shall the common tangents of each of the conics and the ellipse intersect on the directrix.

Solution by the PROPOSER; R. W. GENESE; *and others.*

Project the ellipse into a circle of which S, the focus corresponding to the directrix through the vertex, is the centre. Then, if A, B, C in the original figure become A′, B′, C′ in the projected figure, we have to show that if conics touching A′B′ in O′, the same point as the circle whose centre is S′ touches it, be drawn such that the tangents to any one of them drawn parallel to the pair drawn to the circle through A′ and B′ cut the base in points P′ and Q′ for which the relation O′P′. O′Q′ = O′A′. O′B′ holds, then the common tangents of any such conic and the circle are parallel.

Reciprocate with respect to O′. Then the conics become a series of parabolæ whose axes are parallel, and which are such that a line through $\overline{O}$, the focus of F (that one corresponding to the circle), and cutting it in a and b, cuts any other curve of the system in p and q, so that $\overline{O}p \,.\, \overline{O}q = \overline{O}a \,.\, \overline{O}b$, and we have to show that the chords of intersection of any such parabola and F pass through $\overline{O}$, which is proved by the Rev. J. Wolstenholme's Question 2924.

3037. (Proposed by A. MARTIN.)—A circle is drawn with its centre on the circumference of a given circle: find the average area cut off.

I. *Solution by* STEPHEN WATSON.

It is plain we may take the centre of the variable circle at a *fixed* point in the circumference of the given one. Let a, x be the radii of the given and variable circles; then the area cut off is

$$x^2 \cos^{-1}\left(\frac{x}{2a}\right) + 2a^2 \sin^{-1}\left(\frac{x}{2a}\right) - x\,(a^2 - \tfrac{1}{4}x^2)^{\frac{1}{2}} \quad \text{..............} \quad (1),$$

and x lies between 0 and $2a$; hence the average is

$$\frac{1}{2a}\int_0^{2a} (1)\, dx = a^2\left(\pi - \frac{16}{9}\right).$$

II. *Solution by the* PROPOSER.

1. Let r be the radius of the given circle. With a point C on its circumference as centre, and any radius CB not greater than the diameter of the given circle, draw a circle cutting the given one in D and E, and from the same centre C draw the arc dbe indefinitely near to DBE. Draw also the perpendicular DF.

Let x be the radius of the cutting circle. Then

$$\text{CF} = \frac{x^2}{2r}, \quad \text{arc DBE} = 2x \cos^{-1}\frac{x}{2r}, \quad \text{B}b = dx,$$

and $2x\cos^{-1}\left(\frac{x}{2r}\right)dx$ is the differential of the area cut off. The number of cutting circles is proportional to $2r$.

Hence, if Δ be the average area required, we have

$$\Delta = \frac{1}{2r}\int_0^{2r} dx \int_0^x 2x\cos^{-1}\left(\frac{x}{2r}\right)dx$$

$$= \frac{1}{2r}\int_0^{2r}\left(x^2\cos^{-1}\frac{x}{2r} + 2r^2\sin^{-1}\frac{x}{2r} - \tfrac{1}{2}x\left\{4r^2-x^2\right\}^{\frac{1}{2}}\right)dx$$

$$= r^2\left(\pi - \tfrac{16}{9}\right).$$

2. *Otherwise*:—Let r be the radius of the given circle, and ϕ the angle ACD. Then the angle AOD is 2ϕ, BC is $2r\cos\phi$, the area cut off is $r^2(\pi - \sin 2\phi + 2\phi\cos 2\phi)$, and the average area required is

$$\frac{1}{2r}\int_0^{\frac{1}{2}\pi} r^2(\pi - \sin 2\phi + 2\phi\cos\phi)\, d\,(2r\cos\phi)$$

$$= -r^2\int_0^{\frac{1}{2}\pi}(\pi - \sin 2\phi + 2\phi\cos\phi)\sin\phi\, d\phi$$

$$= -r^2\int_0^{\frac{1}{2}\pi}(\pi\sin\phi - 2\phi\sin\phi - 2\sin^2\phi\cos\phi + 4\phi\sin\phi\cos^2\phi)\, d\phi$$

$$= r^2\left(\pi - \tfrac{16}{9}\right).$$

3086. (Proposed by M. W. Crofton, M.A., F.R.S.)—Prove that

$$e^{\frac{1}{2}D^2}F(x) = e^{-\frac{1}{2}x^2}F(D)\,e^{\frac{1}{2}x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

$$e^{\frac{1}{2}x^2}e^{\frac{1}{2}D^2}e^{\frac{1}{2}x^2}F(x) = e^{\frac{1}{2}D^2}e^{\frac{1}{2}x^2}e^{\frac{1}{2}D^2}F(x). \quad \ldots\ldots\ldots\ldots\ldots (2).$$

Solution by J. J. Walker, M.A.

1. Let ψ be a functional symbol defined by $\psi(x)\,\psi(D)\,F(x) = F(D)\,\psi(x)$, whatever be the subject F. Then, differentiating,

$$\psi'(x)\,\psi(D)\,F(x) + \psi(x)\,D\psi(D)\,F(x) = DFD\psi x = \psi(x)\,\psi(D)\,xF(x) \ldots (1),$$

in virtue of the definition of ψ. But

$$\psi(D)\,xF(x) = x\psi(D)\,F(x) + \psi'(D)\,F(x)$$

(Hargreave, *Phil. Trans.* 1848). Hence (1) becomes

$$\psi'(x)\,\psi(D)\,F(x) + \psi(x)\,D\psi(D)\,F(x) = x\psi(x)\,\psi(D)\,F(x) + \psi(x)\,\psi'(D)\,F(x),$$

or $\quad \left\{\psi'(x) - x\psi(x)\right\}\psi(D)\,F(x) = \psi(x)\left\{\psi'(D) - D\psi(D)\right\}F(x).$

This equation will be satisfied identically if $\psi'(x) - x\psi(x) = 0$; the complete integral of which is $\psi(x) = ce^{\frac{1}{2}x^2}$. The definition requires that $c = 1$.

2. By (1), $$e^{\frac{1}{2}x^2}e^{\frac{1}{2}D^2}e^{\frac{1}{2}x^2}F(x) = e^{\frac{1}{2}D^2}F(D)\,e^{\frac{1}{2}x^2};$$
but again, by (1), $$F(D)\,e^{\frac{1}{2}x^2} = e^{\frac{1}{2}x^2}e^{\frac{1}{2}D^2}F(x);$$
therefore $$e^{\frac{1}{2}x^2}e^{\frac{1}{2}D^2}e^{\frac{1}{2}x^2}F(x) = e^{\frac{1}{2}D^2}e^{\frac{1}{2}x^2}e^{\frac{1}{2}D^2}F(x).$$

[From (1) we may deduce with great ease the value of

$$\frac{d^n}{dx^n}e^{\frac{1}{2}x^2} = e^{\frac{1}{2}x^2}\left(x^n + \frac{n(n-1)}{2}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\,.\,4}x^{n-4} + \&c.\right)\Bigg].$$

3082. (Proposed by J. J. Walker, M.A.)—Let a, b, c, d be any four quantities. Prove that

$$\frac{(ab-cd)(ac-bd)}{(a+b-c-d)(a+c-b-d)} + \frac{(ad-bc)(ac-bd)}{(a+c-b-d)(a+d-b-c)} + \frac{(ab-cd)(ad-bc)}{(a+b-c-d)(a+d-b-c)}$$

$$\equiv \frac{ab(c+d)-cd(a+b)}{a+b-c-d} + \frac{ac(b+d)-bd(a+c)}{a+c-b-d} + \frac{ad(b+c)-bc(a+d)}{a+d-b-c}.$$

Solution by J. A. McNeill.

$$\frac{2(ab-cd)(ac-bd)}{(a+b-c-d)(a+c-b-d)} = \frac{ab(c+d)-cd(a+b)}{a+b-c-d} + \frac{ac(b+d)-bd(a+c)}{a+c-b-d},$$

$$\frac{2(ad-bc)(ac-bd)}{(a+d-b-c)(a+c-b-d)} = \frac{ad(b+c)-bc(a+d)}{a+d-b-c} + \frac{ac(b+d)-bd(a+c)}{a+c-b-d},$$

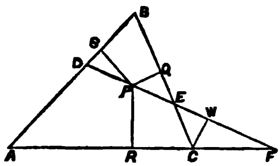
$$\frac{2(ab-cd)(ad-bc)}{(a+b-c-d)(a+d-b-c)} = \frac{ab(c+d)-cd(a+b)}{a+b-c-d} + \frac{ad(b+c)-bc(a+d)}{a+d-b-c};$$

hence, by addition, we obtain the theorem in the question.

3062. (Proposed by A. Renshaw.)—Let ABC be a triangle, and DE a straight line cutting the sides AB, BC in the points D, E, and AC produced towards C in F. Then, if through C a parallel to AB be drawn meeting DE produced in W, and in DE any point P be taken from which PQ, PR, PS are drawn at right angles to the sides BC, CA, AB respectively: prove the following relation:

$$DB\,.\,AC\,.\,PR - AD\,.\,BC\,.\,PQ = CW\,.\,AB\,.\,SP.$$

Solution by J. A. McNeill; *the* Proposer; *and others.*



$BD.BE(\triangle ABC) = AB.BC(\triangle BDE)$,

or $BD.BE(AC.PR+BC.PQ+AB.PS)$
$= AB.BC(BE.PQ+BD.PS)$,

or $BD.BE.AC.PR-AB.BE.BC.PQ$
$+BD.BE.BC.PQ$
$= BD.BC.AP.PS-BD.BE.AB.PS$,

or $BD.BE.AC.PR-AD.BE.BC.PQ$
$= BD.CE.AB.PS$,

or $$BD.AC.PR-AD.BC.PQ = \frac{CE.BD}{BE.AD}.AD.AB.PS$$

$$= \frac{CF}{AF}.AD.AB.PS = \frac{CW}{AD}.AD.AB.PS = CW.AB.PS.$$

[The theorem is, of course, the immediate geometrical signification of the areal equation to a straight line.]

3127. (Proposed by Professor Sylvester.)—Let there be a matrix of two sets of n quantics $X_1, X_2, \ldots\ldots X_n, \Xi_1, \Xi_2, \ldots\ldots \Xi_n$, each containing the same n variables $x_1, x_2, \ldots\ldots x_n$, and of the respective degrees $a_1, a_2, \ldots\ldots a_n$, $\alpha, \alpha_1, \ldots\ldots \alpha_n$, where $a_1-\alpha_1 = a_2-\alpha_2 = \ldots\ldots a_n-\alpha_n = \Delta$. Prove that the number of systems of ratios $x_1 : x_2 : \ldots\ldots : x_n$, which will make all the first minors of the matrix zero, is $\dfrac{a_1a_2\ldots a_n - \alpha\alpha_1\ldots\alpha_n}{\Delta}$.

Solution by Samuel Roberts, M.A.

This is manifestly the same thing as finding the order of conditions for the coexistence of n linear equations

$$\left.\begin{array}{c} \Xi_1 u + X_1 v \\ \Xi_2 u + X_2 v \\ \ldots\ldots\ldots\ldots \\ \Xi_n u + X_n v \end{array}\right\} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(a),$$

the degrees of the coefficients of u and v in the uneliminated variables being $\alpha_1, \alpha_2 \ldots \alpha_n$, and $\alpha_1+\Delta, \alpha_2+\Delta, \ldots\ldots \alpha_n+\Delta$, respectively. The solution of this is given by me in the *Proceedings of the London Mathematical Society*, Vol. II., p. 10, in the form

$$\Delta^{n-1} + \Delta^{n-2}\Sigma\alpha_1 + \Delta^{n-3}\Sigma\alpha_1\alpha_2 + \&c.;$$

and the general result, when the equations are not linear, but of the degrees $m_1, m_2 \ldots m_n$, and the increment of the orders of the coefficients is Δ, is given in precisely the form of the question, viz.

$$\frac{(\Delta m_1+\alpha_1)(\Delta m_2+\alpha_2)\ldots\ldots(\Delta m_n+\alpha_n)-\alpha_1\alpha_2\ldots\alpha_n}{\Delta},$$

the literal notation only being different.

As the formula does not seem to be known, I may reproduce the proof very briefly for the system (a).

Assume that the formula holds for systems of $n-1$ and $n-2$ equations; and consider the system consisting of the system (a), except the second equation, and the system consisting of the first and second equations. The order for these systems jointly is

$$\left\{\frac{a_1 a_3 \dots a_n - a_1 \alpha_3 \dots \alpha}{\Delta}\right\}(a_1+\alpha_2), \quad \text{where } a_1+\alpha_2 = a_2+\alpha_1.$$

But we have included the irrelevant system involving $\Xi_1 = 0$, $X_1 = 0$, and whose order is therefore $a_1 \alpha_1 \left(\dfrac{a_3 a_4 \dots a_n - \alpha_3 \alpha_4 \dots \alpha_n}{\Delta}\right)$.

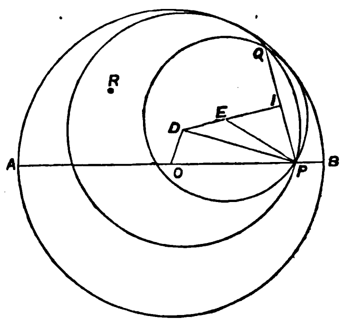
Subtracting this from the preceding order, which may be written in the form

$$\frac{a_1 a_3 \dots a_n (a_2+\alpha_1) - \alpha_1 \alpha_3 \dots \alpha_n (a_1+\alpha_2)}{\Delta},$$

we get the result in the question. But the formula is well known to be true in the case of $n=2$, $n=3$, therefore it holds generally.

1843. (Proposed by the EDITOR.)—Three points being taken at random within a given circle, find the chance that the circle drawn through them shall lie wholly within the given circle.

Solution by STEPHEN WATSON.



Let O be the centre of the given circle; P, Q, R any three points within it; AB a diameter through P; D and E the centres of circles tangential to the given one, and passing through the points P and Q. Join OD, PD, PE, and DE bisecting PQ in I. The required chance will obviously be the same whatever be the length of OB; therefore put OB = 1, OP = x, PQ = y, $\angle$OPQ = ϕ, $\angle$OPD = α, $\angle$OPE = β; and for shortness put $1-x^2 = m$, $1 - x\cos\alpha = X$, $1 - x\cos\beta = X_1$, $1 - x\sin\phi = Y$, $1 + x\sin\phi = Y_1$; then

$$\left\{1 - \tfrac{1}{2}y\sec(\phi-\alpha)\right\}^2 = OD^2 = x^2 + \tfrac{1}{4}y^2\sec^2(\phi-\alpha) - xy\sec(\phi-\alpha)\cos\alpha;$$

therefore $y = \dfrac{m\cos(\phi-\alpha)}{X}$, and by similarity $= \dfrac{m\cos(\phi-\beta)}{X_1}$ ....(1).

In order that the circle through P, Q, R may lie wholly within the given circle, R must lie in one of the lunes formed by the two tangential circles, and the sum of the areas of the two lunes is

$$\tfrac{1}{4}\left\{(\phi-\alpha)\sec^2(\phi-\alpha) - (\phi-\beta)\sec^2(\phi-\beta) + \tan(\phi-\alpha) - \tan(\phi-\beta)\right\}y^2 \dots (2).$$

An element of the circle at P $= 2\pi x\,dx$, at Q $= y\,dy\,d\phi$, and the whole number of positions of P, Q, R is π^3. The limits are x from 0 to 1; ϕ from 0 to $\frac{1}{2}\pi$, and the result doubled, (the case of Q below AB will be considered as we proceed); the lower limit of y is 0, in which case $\alpha = \phi - \frac{1}{2}\pi$, $\beta = \phi + \frac{1}{2}\pi$; and the upper limit of y is when Q is on the circumference of the given circle, in which case $\alpha = \beta$, and from the symmetry of (1) and (2) it is plain that the integral of (2) will vanish at this limit of y, and may therefore be altogether disregarded.

The limits thus understood, the required chance (p) may be written

$$\frac{2}{\pi^3}\iiint\left\{(\phi-\alpha)\sec^2(\phi-\alpha)-(\phi-\beta)\sec^2(\phi-\beta)+\tan(\phi-\alpha)-\tan(\phi-\beta)\right\}$$
$$\times\, y^3 x\,dx\,d\phi\,dy \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Now $$2\int\sec^2(\phi-\alpha)\,y^3dy = 2m^4\int\frac{\cos(\phi-\alpha)\left\{\sin(\phi-\alpha)-x\sin\phi\right\}}{X^5}\,d\alpha$$

$$= m^4\int\left\{\frac{\sin 2\phi}{X^5}\left[\frac{2(1-X)^2}{x^2}-1-(1-X)\right]d\alpha\right.$$
$$\left.-\frac{\cos 2\phi}{X^5}\left[\frac{2(1-X)}{x}-x\right]\frac{dX}{x}-\frac{dX}{X^5}\right\}$$

$$= m^4\int\left\{\frac{\sin 2\phi}{x^2}\left[\frac{2m}{X^5}-\frac{4-x^2}{X^4}+\frac{2}{X^3}\right]d\alpha-\frac{\cos 2\phi}{x^2}\left[\frac{2-x^2}{X^5}-\frac{2}{X^4}\right]dX-\frac{dX}{X^5}\right\}$$

$$= m^4\left\{\frac{\sin 2\phi\sin\alpha}{12x}\left[\frac{6}{X^4}-\frac{2(1-2x^2)}{mX^3}-\frac{2-7x^2}{m^2X^2}-\frac{2-9x^2-8x^4}{m^3X}\right]\right.$$
$$+\frac{5x^2}{2m^{\frac{3}{2}}}\sin 2\phi\tan^{-1}\left[\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}\alpha\right]$$
$$\left.+\frac{\cos 2\phi}{x^2}\left[\frac{2-x^2}{4X^4}-\frac{2}{3X^3}\right]+\frac{1}{4X^4}\right\}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

(See Gregory's *Examples*, pp. 281 and 255.)

Integrating now "by parts," we have

$$2\int(\phi-\alpha)\sec^2(\phi-\alpha)\,y^3dy = (4)(\phi-\alpha)+\int(4)\,d\alpha$$

$$= (4)(\phi-\alpha)-\frac{m^4\sin 2\phi}{12x^2}\left\{\frac{2}{X^3}-\frac{1-2x^2}{mX^2}-\frac{2-7x^2}{m^2X}+\frac{2-9x^2-8x^4}{m^3}\log X\right\}$$
$$+\frac{m^3\cos 2\phi}{24x^2}\left\{x\sin\alpha\left[\frac{2(2-x^2)}{X^3}+\frac{2+3x^2}{mX^2}-\frac{2-21x^2+4x^4}{m^2X}\right]\right.$$
$$\left.-\frac{8-40x^2+2x^4}{m^{\frac{3}{2}}}\tan^{-1}\left[\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}\alpha\right]\right\}$$
$$+\frac{m^3}{24}\left\{x\sin\alpha\left[\frac{2}{X^3}+\frac{5}{mX^2}+\frac{11+4x^2}{m^2X}\right]\right.$$
$$\left.+\frac{12+18x^2}{m^{\frac{3}{2}}}\tan^{-1}\left[\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}\alpha\right]\right\}$$
$$+\frac{5}{2}\int m\;x^2\sin 2\phi\tan^{-1}\left[\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}\alpha\right]d\alpha\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

By a like process we have

$$2\int \tan(\phi-\alpha)\,y^3dy = 2m^4\int \frac{\sin(\phi-\alpha)\cos^3(\phi-\alpha)\left\{\sin(\phi-\alpha)-x\sin\phi\right\}}{X^5}\,d\alpha$$

$$=\frac{m^4}{4}\int\left\{\frac{d\alpha}{X^4}-\frac{\cos 4\phi}{x^4}\left[\frac{4(2-x^2)(1-x^2)}{X^5}-\frac{32-28x^2+3x^4}{X^4}+\frac{48-20x^2}{X^3}\right.\right.$$
$$\left.-\frac{32-4x^2}{X^2}+\frac{8}{X}\right]d\alpha$$
$$-\frac{\sin 4\phi}{x^4}\left[\frac{8(1-x^2)+x^4}{X^5}-\frac{24-12x^2}{X^4}+\frac{24-4x^2}{X^3}-\frac{8}{X^2}\right]dX$$
$$+\frac{4\cos 2\phi}{x^2}\left[\frac{1-x^2}{X^5}-\frac{3-x^2}{X^4}+\frac{3}{X^3}-\frac{1}{X^2}\right]d\alpha$$
$$\left.+\frac{2\sin 2\phi}{x^2}\left[\frac{2-x^2}{X^5}-\frac{4}{X^4}+\frac{2}{X^3}\right]dX\right\}$$

$$=\frac{m^4}{4}\left\{\frac{x\sin\alpha}{6}\left[\frac{2}{mX^3}+\frac{5}{m^2X^2}+\frac{11+4x^2}{m^3X}\right]+\frac{2+3x^2}{m^{\frac{7}{2}}}\tan^{-1}\left[\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}\alpha\right]\right.$$
$$-\frac{\cos 4\phi}{x^4}\left[x\sin\alpha\left(\frac{2-x^2}{X^4}-\frac{6-x^2}{X^3}+\frac{6}{X^2}-\frac{2}{X}\right)\right]$$
$$+\frac{\sin 4\phi}{x^4}\left[\frac{8(1-x^2)+x^4}{4X^4}-\frac{8-4x^2}{X^3}+\frac{12-2x^2}{X^2}-\frac{8}{X}\right]$$
$$+\frac{\cos 2\phi}{x^2}\left[x\sin\alpha\left(\frac{1}{X^4}-\frac{5-4x^2}{3mX^3}+\frac{2-7x^2}{6m^2X^2}+\frac{2-9x^2-8x^4}{6m^3X}\right)\right.$$
$$\left.-\frac{5x^4}{m^{\frac{7}{2}}}\tan^{-1}\left[\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}\alpha\right]\right]$$
$$\left.-\frac{\sin 2\phi}{x^2}\left[\frac{2-x^2}{2X^4}-\frac{8}{3X^3}+\frac{2}{X^2}\right]\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

The variable being understood as changed to α or β, as the case may require, in the manner above, it is plain that

$$2\int_{\phi-\frac{1}{2}\pi}\left\{(\phi-\alpha)\sec^2(\phi-\alpha)+\tan(\phi-\alpha)\right\}y^3dy$$
$$-2\int_{\phi+\frac{1}{2}\pi}\left\{(\phi-\beta)\sec^2(\phi-\beta)+\tan(\phi-\beta)\right\}y^3dy=\left\{(5)+(6)\right\}_{\phi-\frac{1}{2}\pi}^{\phi+\frac{1}{2}\pi}.$$

For shortness, put

$$t_1=\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}(\phi+\tfrac{1}{2}\pi),\quad t_2=\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}(\phi-\tfrac{1}{2}\pi),$$

and t_3, t_4 for what t_1, t_2 respectively become when $\pi-\phi$ is put for ϕ. Taking the part of (4) $(\phi-\alpha)$ in (5) affected with $\tan^{-1}$, between the above limits, putting $\pi-\phi$ for ϕ, to include the case of Q below AB, and adding the results, we get

$$-\tfrac{5}{4}\pi m^{\frac{1}{2}}x^2\sin 2\phi\,(\tan^{-1}t_1+\tan^{-1}t_2-\tan^{-1}t_3-\tan^{-1}t_4)\;\ldots\ldots\ldots(7).$$

Multiplying by $d\phi$, and integrating "by parts" the portion affected with ϕ, we have

$$\sin^2\phi\,(\tan^{-1}t_1+\tan^{-1}t_2-\tan^{-1}t_3-\tan^{-1}t_4)-2m^{\frac{1}{2}}\int\frac{\sin^2\phi\,d\phi}{1-x^2\sin^2\phi}.$$

Whence we easily find

$$\int_0^1 x\,dx\int_0^{\frac{1}{2}\pi}(7)\,d\phi=-\tfrac{5}{4}\pi^2\int_0^1\left\{(1-x^2)-(1-x^2)^{\frac{3}{2}}\right\}x\,dx=\tfrac{5}{48}\pi^2\;\ldots\;(8).$$

Take now the remaining portion of (4) $(\phi-\alpha)$ in (5) at the upper limit $\alpha=\phi+\frac{1}{2}\pi$; then put $\pi-\phi$ for ϕ, and add the results, we have

$$-m^4\pi\left\{\frac{\sin 2\phi\cos\phi}{12x}\left[\frac{6}{Y_1^4}-\frac{2(1-2x^2)}{mY_1^3}-\frac{2-7x^2}{m^2Y_1^2}-\frac{2-9x^2-8x^4}{m^3Y_1}\right]\right.$$
$$\left.+\frac{\cos 2\phi}{x^2}\left[\frac{2-x^2}{4Y_1^4}-\frac{2}{3Y_1^3}\right]+\frac{1}{4Y_1^4}\right\}\;\ldots\ldots\ldots(9).$$

But $$\frac{\sin 2\phi\cos\phi}{x}=\frac{2}{x^4}\left\{(Y_1-1)\,x^2-(Y_1-1)^3\right\},$$

and $$\frac{\cos 2\phi}{x^2}=\frac{1}{x^4}\left\{x^2-2(Y_1-1)^2\right\};$$

hence, by substitution, (9) reduces to

$$-\frac{m^2\pi}{6}\left(\frac{m}{Y_1^2}+\frac{5}{Y_1}\right)+\frac{m\pi}{6x^2}\left\{1+3x^2-4x^4-15x^3\sin\phi-(2-9x^2-8x^4)\sin^2\phi\right\}$$
$$\ldots\ldots\ldots(10).$$

By taking the same portion of (4) $(\phi-\alpha)$ in (5) at the lower limit $\alpha=\phi-\frac{1}{2}\pi$, and proceeding in the same way, we get

$$-\frac{m^2\pi}{6}\left(\frac{m}{Y^2}+\frac{5}{Y}\right)+\frac{m\pi}{6x^2}\left\{1+3x^2-4x^4+15x^3\sin\phi-(2-9x^2-8x^4)\sin^2\phi\right\}$$
$$\ldots\ldots\ldots(11).$$

Restoring the values of Y, Y_1, we easily find

$$\int_0^1 x\,dx\int_0^{\frac{1}{2}\pi}\left\{(10)+(11)\right\}d\phi=\tfrac{9}{80}\pi^2\;\ldots\ldots\ldots\ldots\ldots\ldots(12).$$

Again, multiplying the last line of (5) by $x\,dx$, and integrating "by parts," with respect to x, it becomes

$$-\int\left\{(\tfrac{5}{8}m^{\frac{3}{2}}x^2+\tfrac{1}{8}m^{\frac{5}{2}})\sin 2\phi\tan^{-1}\left[\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}\alpha\right]\right\}d\alpha$$
$$+\iint(\tfrac{5}{12}m^2x^2+\tfrac{1}{6}m^3)\sin 2\phi\,\frac{\sin\alpha}{X}\,dx\,d\alpha\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13).$$

But $$\int\sin 2\phi\,\frac{\sin\alpha\,d\alpha}{X}=\frac{1}{x}\sin 2\phi\log X;$$

and if this be taken between the above limits of α, and $\pi-\phi$ be then put for ϕ, the results when added give zero; hence the second line in (13) vanishes. The first line of (13) taken between $x=0$, $x=1$, becomes

$$\int\tfrac{1}{6}\sin 2\phi\,\alpha\,d\alpha=\tfrac{1}{12}\sin 2\phi\,\alpha^2;$$

and this between $\alpha = \phi - \frac{1}{2}\pi$ and $\alpha = \phi + \frac{1}{2}\pi$, then putting $\pi - \phi$ for ϕ, and adding, becomes
$$\tfrac{1}{8}\pi \sin 2\phi \, (2\phi - \pi) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14),$$
and
$$\int_0^{\frac{1}{2}\pi} (14)\, d\phi = -\tfrac{1}{12}\pi^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15).$$

Take now the remaining portion of (5) not yet integrated, and also (6) between the above limits of α, and then put $\pi - \phi$ for ϕ, and add; all those terms not affected with $\tan^{-1}$ will be found to vanish, and we have
$$-\frac{m^{\frac{1}{2}}}{6x^2}\left\{2m(1-4x^2)\cos 2\phi - x^2(6+9x^2)\right\}(\tan^{-1}t_1 - \tan^{-1}t_2 + \tan^{-1}t_3 - \tan^{-1}t_4) \quad\ldots\ldots(16).$$

Multiplying the part of this affected by $\cos 2\phi$, by $d\phi$, and integrating in the same way as (7) was integrated, the result vanishes. Again, multiplying the remaining portion of (16) by $x\,dx$, and integrating "by parts" with respect to x, the result is simply
$$-m^{\frac{3}{2}}\left(\tfrac{1}{3} + \tfrac{1}{2}x^2 + \tfrac{1}{5}m\right)(\tan^{-1}t_1 - \tan^{-1}t_2 + \tan^{-1}t_3 - \tan^{-1}t_4) \quad\ldots\ldots(17),$$
since the coefficient of dx vanishes; and taking (17) between $x = 0$, $x = 1$, it becomes $\frac{8}{15}\pi$; and
$$\tfrac{8}{15}\pi \int_0^{\frac{1}{2}\pi} d\phi = \tfrac{4}{15}\pi^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18).$$

Finally, by (8), (12), (15), and (18), we see that (3) becomes
$$p = \frac{1}{\pi^2}\left(\frac{5}{48} + \frac{9}{80} - \frac{1}{12} + \frac{4}{15}\right)\pi^2 = \frac{2}{5}.$$

[This solution is effected by a direct and entirely satisfactory method; so that it is now thoroughly settled that $\frac{2}{5}$ is the true answer to the question. Mr. Watson's investigation of the problem does him much credit. The work after (16) may be somewhat simplified by observing that
$$\tan^{-1}t_1 - \tan^{-1}t_2 + \tan^{-1}t_3 - \tan^{-1}t_4 = \pi.]$$

3129. (Proposed by the Rev. G. H. Hopkins, M.A.)—A variable circle with its centre upon the circumference of a fixed circle passes through a fixed point on the same; required a geometrical proof that its envelope is a cardioid. Prove also the converse, that a circle passing through the pole of a cardioid and touching the curve has its centre on another circle, which also passes through the pole.

I. *Solution by* Professor Hirst.

The method of quadric, or rather cyclic, inversion, of which a description has been given in the *Reprint*, Vol. I., p. 44, and in the *Proceedings of the Royal Society*, adapts itself readily to the solution of questions of this character.

The envelope in question may obviously be also defined to be that of a circle whose diameter is a chord, of a fixed circle, drawn from a fixed point in its circumference. This fixed point being taken as the origin O, the

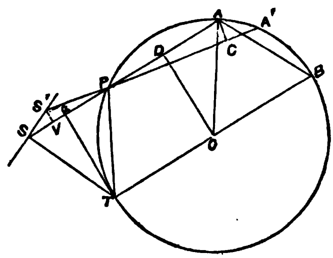
inverse of the fixed circle is a fixed straight line L, and that of the variable circle is a variable straight line T, at right angles to the line which connects its intersection with L and the origin O.

Now the envelope of this line T is well known to be a parabola of which O is the focus and L the tangent at the vertex. But by the principles of Quadric Inversion, the inverse of this parabola—which is of course the required envelope—is a quartic curve having a cusp at the origin and two other cusps at the circular points at infinity,—in other words, it is a cardioid.

In consequence of the mutual relations between inverse curves, the converse problem requires no special consideration.

[The cardioid may perhaps be better known to some readers as the pedal of a circle with respect to an origin on its circumference. Now every pedal is the inverse of the reciprocal of the primitive curve; and the reciprocal of a circle, with respect to another circle having its centre on the circumference of the first, is well known to be a parabola whose focus is at the origin. Hence the cardioid is the focal inverse of a parabola.]

II. *Solution by* J. J. Walker, M.A.



1. Let O be the centre of, P the fixed point on, the circle on the circumference of which lies the centre of the variable circle passing through P, the envelope of which is sought. Let T be one position of the centre; then, drawing the tangent TQ, meeting a perpendicular from P in Q, and taking QS = QP, it is plain that S will be the second point of intersection of the variable circle with its consecutive circle. But $AQ . QS = AQ . QP = TQ^2$, and ATS is therefore a right angle, and ST parallel to AB; therefore ASTB is a parallelogram, and AS = BT. The locus of S is therefore a cardioid.

2. The circle in the figure being now supposed to be the generating circle of a cardioid, of which P is pole and S a point, draw the diameter BT parallel to PS, and the tangent TQ at T. It follows at once that PQ = QS. Let S′ be the consecutive point on the cardioid; draw AC perpendicular to A′PS′, and S′V, OD to AS. The triangles AOD, A′CA are equiangular, therefore $\frac{AD}{OD}$, *i.e.* $\frac{AP}{PS} \cdot \frac{SQ}{QT} = \frac{AC}{CA'} = \frac{AC}{SV} = \frac{AP}{PS} \cdot \frac{S'V}{SV}$.

Hence $\frac{SQ}{QT} = \frac{S'V}{SV}$, and the triangles SQT, S′VS are equiangular, and ST is therefore perpendicular to SS′, the tangent to the cardioid at S. The circle described with T as centre and TP as radius will therefore touch the cardioid at S, since TS = TP; therefore &c.

3116. (Proposed by the Rev. E. Hill, M.A.)—Along the edge of an elliptic lamina a weight slides perfectly freely. The lamina is set floating

in fluid with its plane vertical. Show that if it can rest in one position with neither axis vertical, then it will rest in all positions; and that if it be turned round so as to pass through all such positions, the dry part will be a similar ellipse.

Solution by C. W. MERRIFIELD, F.R.S.

The proposition is an easy deduction from a proposition relating to the ellipsoid and to quadric surfaces generally, stated in Scott Russell's *Modern Naval Architecture*, published in 1865.

If a homogeneous ellipsoid float in various positions, the locus of its centre of buoyancy and the envelope of its planes of flotation are concentric, similar, and similarly situated ellipsoids. For this is true of a sphere, and these are projective properties.

Again, the points at which the tangent planes are parallel for the bounding surface and the surface of buoyancy, lie on the same radius vector, which is always divided by the latter surface in a constant ratio. Hence, if the upward pressure of the water and the downward pressure of the weight of the ellipsoid be balanced in any one inclined position by a sliding weight, which falls to the lowest point, it will be so balanced always.

Mr. Hill's question is this proposition stated for two dimensions. The question of the sliding weight is, I believe, quite new.

2991. (Proposed by S. WATSON.)—Through the extremities of the major axis of an ellipse, two lines are drawn in a random direction; what is the chance of their intersecting within the ellipse?

Solution by ARTEMAS MARTIN.

Let a, b be the semi-axes of the ellipse, D the centre, and AB the major axis. Draw a line through A meeting the ellipse in C, and making an angle θ with AB. Join BC, and put the angle CBD $=\phi$. Then, if the line drawn through B is above AB, and makes with AB an angle less than ϕ, it will intersect the line through A and C within the ellipse, the chance of which is $\frac{\phi}{\pi}$. Draw CD perpendicular to AB; then, by a property of the ellipse, we have $a^2 : b^2 = \text{AD} \,.\, \text{DB} : \text{CD}^2 = 1 : \tan\theta\tan\phi$;

therefore $$\phi = \tan^{-1}\left(\frac{b^2}{a^2}\cot\theta\right).$$

Hence the chance required is

$$p = \frac{2}{\pi^2}\int_0^{\frac{1}{2}\pi}\tan^{-1}\left(\frac{b^2}{a^2}\cot\theta\right)d\theta.$$

I have not found any satisfactory method of calculating the value of this integral. [When $a=b$, the ellipse is a circle, and the chance is $\frac{1}{4}$.]

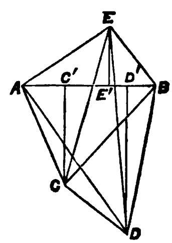
3032. **(Proposed by M. W. Crofton, F.R.S.)—All circles which touch two given circles are cut orthogonally by the pair of circles which pass through the intersections of the given ones, and bisect their angles.**

Solution by the Rev. R. Townsend, M.A., F.R.S.; S. Roberts, M.A.; R. W. Genese; *and others.*

This property, which is evident by inversion from either point of intersection of the two given circles, is a particular case of the more general property (see *Modern Geometry*, Art. 209), that the two systems of circles which touch, one similarly and the other oppositely, two given circles, are cut orthogonally, the former and the latter respectively, by the two circles of antisimilitude, external and internal, of the two given circles, which respectively, when the latter intersect, bisect, externally and internally, their two angles of intersection.

3002. (Proposed by Matthew Collins, B.A.)—If every two of five circles A, B, C, D, E touch each other, except D and E, prove that the common tangent of D and E is just twice as long as it would be if D and E touched each other.

Solution by Asher B. Evans, M.A.



Let A, B, C, D, E be the centres of the five circles, and a, b, c, d, e their radii. Draw CC′, DD′, EE′ perpendicular to AB. Let m be the common tangent to D and E as now situated, and m_1 their common tangent when they are in contact. Then

$$(AD'-AE')^2+(DD'+EE')^2-(d-e)^2=m^2 \;\ldots\; (1),$$

$$(d+e)^2-(d-e)^2=4de=m_1^2 \;\ldots\ldots\ldots\ldots\; (2),$$

$$(AD'-AC')^2+(DD'-CC')^2=CD^2=(c-d)^2 \;\ldots\; (3),$$

$$(AE'-AC')^2+(EE'-CC')^2=CE^2=(c-e)^2 \;\ldots\; (4).$$

From the geometry of the figure,

$$AC'=a-\left(\frac{a-b}{a+b}\right)c,\quad AD'=a+\left(\frac{a-b}{a+b}\right)d,\quad AE'=a+\left(\frac{a-b}{a+b}\right)e,$$

$$CC'=\frac{2\,(ab)^{\frac12}}{a+b}\left\{c^2-c(a+b)\right\}^{\frac12},\quad DD'=\frac{2\,(ab)^{\frac12}}{a+b}\left\{d\;+d(a+b)\right\}^{\frac12},$$

$$EE'=\frac{2\,(ab)^{\frac12}}{a+b}\left\{e^2+e(a+b)\right\}^{\frac12} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (5).$$

By the aid of (5) we may reduce (1), (3), (4) to (6), (7), (8),

$$\frac{8ab}{(a+b)^2}\left\{(a+b)\frac{d+e}{2de}+1+\left(\frac{(a+b)^2}{de}+(a+b)\frac{d+e}{de}+1\right)^{\frac12}\right\}de=m^2 \;\ldots\; (6),$$

$$\left\{(a-b)^2c^2+2abc\,(a+b)+a^2b^2\right\}d^2-2abc\,(ac+bc-ab)\,d+a^2b^2c^2=0 \;\ldots\; (7),$$

$$\left\{(a-b)^2c^2+2abc\,(a+b)+a^2b^2\right\}e^2-2abc\,(ac+bc-ab)\,e+a^2b^2c^2=0 \;\ldots\; (8).$$

By the theory of equations, we obtain from (7) and (8)

$$d+e = \frac{2abc\,(ac+bc-ab)}{(a-b)^2 c^2 + 2abc\,(a+b) + a^2b^2}, \quad de = \frac{a^2b^2c^2}{(a-b)^2 c^2 + 2abc\,(a+b) + a^2b^2}.$$

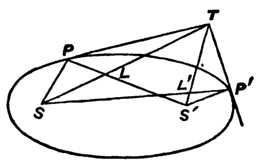
By substituting these values of $(d+e)$ and de in (6), we reduce it to

$$16de = m^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9).$$

Therefore, from (2) and (9), $\qquad m = 2m_1$.

3080. (Proposed by C. TAYLOR, M.A.)—The straight line joining the foci of a conic subtends at the pole of any chord half the sum or difference of the angles which it subtends at the extremities of the chord.

Solution by the Rev. J. WOLSTENHOLME, M.A.



Let TP, TP′ be tangents, S, S′ the foci; and let ST, S′P meet in L, S′T, SP′ in L′; then $\angle$ SPS′ = PLT − PST = PLT − ½PSP′; also $\angle$ STS′ = PLT − PS′T = PLT − ½PS′P′, or $\angle$ SPS′ − STS′ = ½ (PS′P′ − PSP′). So $\angle$ STS′ − SP′S′ = ½ (PS′P′ − PSP′); or $\angle$ SPS′ − STS′ = STS′ − SP′S′; or $\angle$ STS′ is the semi-sum of the angles SPS′, SP′S′. So, if PP′ crosses SS′, the angle STS′ is half the difference of the angles SPS′, SQS′.

3081. (Proposed by B. W. HORNE, M.A.)—Rays diverging from the pole of the cardioid $r = a\,(1-\cos\theta)$ are reflected at the curve; show that the length of the caustic is $6a$.

Solution by MORGAN JENKINS, M.A.

The radio-tangential angle of the cardioid is $\frac{1}{2}\theta$; therefore the angle made with the initial line by the reflected ray is $\theta - 2\left(\frac{1}{2}\pi - \frac{1}{2}\theta\right)$ or $2\theta - \pi$.

The equations giving, by the elimination of θ, the equation of the caustic, are

$$x\sin 2\theta - y\cos 2\theta = a\left(\sin\theta - \tfrac{1}{2}\sin 2\theta\right) \quad\ldots\ldots\ldots\ldots \quad (1),$$

$$x\cos 2\theta + y\sin 2\theta = \tfrac{1}{2}a\left(\cos\theta - \cos 2\theta\right) \quad\ldots\ldots\ldots\ldots \quad (2);$$

and differentiating these equations on the supposition that x and y are functions of θ, we have

$$\frac{dx}{d\theta}\sin 2\theta - \frac{dy}{d\theta}\cos 2\theta = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

and

$$\frac{dx}{d\theta}\cos 2\theta + \frac{dy}{d\theta}\sin 2\theta + 2\left(y\cos 2\theta - x\sin 2\theta\right) = \tfrac{1}{2}a\left(2\sin 2\theta - \sin\theta\right)\ldots(4).$$

From these four equations we obtain $\frac{dx}{d\theta}\sec 2\theta = \frac{3}{2}a\sin\theta$.

But if σ be the length of the caustic,

$$\frac{d\sigma}{d\theta} = \left\{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right\}^{\frac{1}{2}} = \frac{dx}{d\theta}\sec 2\theta\,;$$

therefore $$\sigma = 2\int_0^{\pi} \tfrac{3}{2}a\sin\theta = (3a\cos\theta)_{\pi}^{0} = 6a.$$

2570. (Proposed by R. Tucker, M.A.)—Find the number of ways in which the first 9 digits may be arranged so as to make up 99.

Solution by the Proposer.

Let x = sum of digits in *tens* line, and y = sum of digits in *units* line; then $\left.\begin{aligned} x+y &= 45 \\ 10x+y &= 99 \end{aligned}\right|$; therefore $x = 6$.

α	β	γ	δ
6	5	4	3
	1	2	2
			1

$$\left.\begin{array}{lr} \alpha \text{ gives} & 8 \\ \beta \quad ,, & 42 \\ \gamma \quad ,, & 42 \\ \delta \quad ,, & 120 \end{array}\right| = 212 \text{ ways.}$$

3124. (Proposed by Sir James Cockle, F.R.S.)—Boole remarked (*Differential Equations*, 2nd Ed., p. 362, Art. 1, and p. 380, Ex. 7) that Monge's method would not enable us to solve the equation $r - t = \frac{2p}{x}$ ... (1).

Solve the above and the more general equation $r - a^2t = \frac{2np}{x}$ ....(2),

where n is an integer; and find a case in which $r - t = \frac{2np}{x}$ .... (3) is soluble by Monge's method.

I. *Solution by* J. J. Walker, M.A.

1. The solution of (2) is given in Gregory's *Examples*, Ch. VI., Ex. 23, in the form $$z = x^{n+1}\left(\frac{d}{dx}\right)^n \frac{1}{x}\left\{F(y+ax) + f(y-ax)\right\}.$$

When $n=1$, $a=1$, this gives as the solution of (1)

$$z = \left(x\frac{d}{dx} - 1\right)\left\{F(y+x) - f(y-x)\right\}.$$

Otherwise, by assuming $t = x^{2n+1}$, $a' = \frac{a}{2n+1}$, equation (2) is transformed into

$$t^{\frac{4n}{2n+1}} \frac{d^2z}{dt^2} - a'^2 \frac{d^2z}{dy^2} = 0.$$

The solution of $t^{\frac{4n}{2n+1}} \frac{d^2z}{dt^2} - a'^2 z = 0$ is given by Euler (*Calc. Int.*, § 952); and if, in this equation, a' be replaced by $a' \frac{d}{dy}$, the arbitrary constants α, β by arbitrary functions of y, and finally t be replaced by x^{2n+1}, the solution of (2) will be obtained.

2. Trying, by Monge's method, $r - a^2t = \frac{2np}{x}$,

$$dy - adx = 0 \;\ldots\ldots\ldots (1), \qquad dp - adq = \frac{2n}{x} pdx \;\ldots\ldots\ldots (2),$$

or
$$x(dp - adq) = 2npdx.$$

Adding to both sides $(p-aq)\,dx$, $d(p-aq)\,x = (2n+1)\,pdx - aqdx$, the second side of which will equal $-dz$, if $2n+1 = -1$, *i.e.* if $n = -1$. Integrating, $(p-aq)\,x + z = c'$, while from (1) $y - ax = c$; consequently $(p-aq)\,x + z = f(y-ax)$ is a first integral. To integrate this by Lagrange's method,

$$\frac{dx}{x} = -\frac{dy}{ax} = \frac{dz}{f(y-ax)-z}.$$

The first two give $y + ax = k$; hence the first and third

$$\frac{dx}{x} = \frac{dz}{f(k-2ax)-z}, \text{ or } x\frac{dz}{dx} + z = f(k-2ax),$$

whence
$$z = \frac{1}{x}\left\{\int f(y-ax)\,dx + k'\right\} \text{ or } xz - \int f(k-2ax)\,dx = k'.$$

The complete integral is therefore

$$xz - \int f(k-2ax)\,dx = \psi(y+ax)$$

[k being replaced, after integration, by $y+ax$],

or
$$z = \frac{1}{x}\left\{\int f(k-2ax)\,dx + \psi(y+ax)\right\},$$

or simply
$$z = \frac{1}{x}\left\{\chi(y-ax) + \psi(y+ax)\right\}.$$

The case in which (3) is soluble by Monge's method, then, is when $n = -1$, besides the well-known case of $n = 0$.

II. *Solution by the* Proposer.

Firstly. The equations comprised in the formula

$$p \mp q = f(y \pm x) = f \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

although they do not constitute a system of Mongian first integrals of (1), do, in a certain sense, constitute a system of first integrals, inasmuch as

the solutions of (1) are contained among those of (4). From (4) we derive

$$r \mp s = \frac{df}{dx}, \quad s \mp t = \frac{df}{dy} = \pm \frac{df}{dx} \text{ ....(5, 6)};$$

whence

$$r - t = 2\frac{df}{dx} \text{ ....(7)};$$

and (7) will be identical with (1) provided that

$$\frac{df}{dx} = \frac{p}{x} = \frac{1}{x}\frac{dz}{dx}, \text{ or } \frac{dz}{dx} = x\frac{df}{dx} \text{ ....(8, 9)}.$$

Integrating (9), we have

$$z = xf(y \pm x) - \int f(y \pm x)\,dx = xf(y \pm x) \mp \int f(y \pm x)(dy \pm dx) \text{ ......(10)}.$$

Moreover the solution of (4) is

$$z = xf(y \pm x) + \phi(y \pm x) \text{ ....(11)},$$

where ϕ is an arbitrary function. Hence, comparing (10) with (11), we find

$$\phi(y \pm x) = \mp \int f(y \pm x)(dy \pm dx) \text{ ....(12)},$$

or

$$\frac{d\phi(y \pm x)}{d(y \pm x)} = \phi'(y \pm x) = \mp f(y \pm x) \text{ ....(13)}.$$

Hence, by means of (12) and (13), (11) may be put under the form

$$z = \phi(y \pm x) \mp x\phi'(y \pm x) \text{ ....(14)}.$$

But (1) is linear, and consequently (14) enables us to write its general solution in the form

$$z = \phi(y+x) + \psi(y-x) - x\left\{\phi'(y+x) - \psi'(y-x)\right\} \text{ ....(15)},$$

which is identical with equation (4) of p. 362 of Boole. See also pp. 459 and 496.

Secondly. To solve (2), write it in the form

$$\frac{d^2z}{dx^2} - \frac{2n}{x}\frac{dz}{dx} - a^2\frac{d^2z}{dy^2} = 0 \text{ ....}(a).$$

Next suppose that

$$z = x^n u \text{ ....}(b),$$

and in (a) substitute for z its value as given by (b). Then (a) becomes

$$x^n\frac{d^2u}{dx^2} + \left\{n(n-1) - 2n^2\right\}x^{n-2}u - a^2x^n\frac{d^2u}{dy^2} = 0 \text{ .... }(c);$$

or, reducing and dividing by x^n,

$$\frac{d^2u}{dx^2} - \frac{n(n+1)}{x^2}u - a^2\frac{d^2u}{dy^2} = 0 \text{ ....}(d).$$

Replacing n by i, and changing the order of the terms, (d) becomes

$$\frac{d^2u}{dx^2} - a^2\frac{d^2u}{dy^2} - \frac{i(i+1)}{x^2}u = 0 \text{ ....}(e),$$

which last equation is solved as Ex. 9 of p. 425 of Boole.

Thirdly. The Mongian auxiliary equations for the solution of (3) are (See Boole, pp. 367 and 369)

$$dy^2 - dx^2 = 0, \quad \text{and} \quad dp\,dy - dq\,dx - \frac{2np}{x}dx\,dy = 0 \text{ ....}(f, g),$$

and (f) gives $$dy = \pm dx \quad \ldots\ldots (h);$$
whence (g) becomes, after substitution and division by dy,
$$dp \mp dq - \frac{2np}{x} dx = 0 \quad \ldots\ldots (i),$$
or
$$\begin{aligned} 0 &= x(dp \mp dq) - 2np\,dx \\ &= x(dp \mp dq) + (p \mp q)\,dx - (2n+1)\,pdx \pm qdx \\ &= d\left\{x(p \mp q)\right\} + dz - 2(n+1)\,pdx \quad \ldots\ldots (j); \end{aligned}$$
for, in virtue of (h), $dz = pdx + qdy = pdx \pm qdx$ ......(k)

Let $n+1 = 0$, or $n = -1$; then, integrating (j), we have
$$z + x(p \mp q) = a \quad \ldots\ldots (l),$$
and from (h) we obtain $$y \mp x = b \quad \ldots\ldots (m).$$
Hence two first integrals of $r - t = -\dfrac{2p}{x}$ ......(n)
are embraced in the equation
$$z + x(p \mp q) = f(y \mp x) \quad \ldots\ldots (o),$$
wherein the function f is arbitrary.

3142. (Proposed by R. W. GENESE.)—At the middle points A′, B′, C′ of the sides of a triangle ABC draw perpendiculars $A'\alpha$, $B'\beta$, $C'\gamma$ to those sides all outwards or all inwards, and respectively proportional to them: then the centre of gravity of the triangle $\alpha\beta\gamma$ will coincide with the centre of gravity of ABC.

I. *Solution by the* REV. G. H. HOPKINS, M.A.

If $\lambda a, \lambda b, \lambda c$ be the lengths of the perpendiculars, the distances of α, β, γ from the sides a, b, c will be respectively
$$-\lambda a, \quad \lambda b \cos C + \tfrac{1}{2} b \sin C, \quad \lambda c \cos B + \tfrac{1}{2} c \sin B;$$
the distance of the centre of gravity of the triangle $\alpha\beta\gamma$ from the side a will be $\tfrac{1}{3}\left\{-\lambda a + \lambda b \cos C + \tfrac{1}{2} b \sin C + \lambda c \cos B + \tfrac{1}{2} c \sin B\right\}$,
or $\tfrac{1}{3}\lambda(-a + b\cos C + c\cos B) + \tfrac{1}{3}(\tfrac{1}{2} b \sin C + \tfrac{1}{2} c \sin B)$,
which is $\tfrac{1}{3}(\tfrac{1}{2} b \sin C + \tfrac{1}{2} c \sin B)$ or $\tfrac{1}{3} b \sin C$.

In the same way, the distances from the other sides will be $\tfrac{1}{3} a \sin B$ and $\tfrac{1}{3} c \sin A$, the same as the centre of gravity ABC.

II. *Solution by the* REV. R. TOWNSEND, M.A., F.R.S.; *the* PROPOSER; *and others.*

The centre of gravity of a triangle being that of three equal masses situated either at its three vertices, or at the middle points of its three

sides, and three lines from the middle points of the three sides, perpendicular and proportional to their three lengths, representing three forces in equilibrium, the property in question is consequently a particular case of the following:—

If from a system of any number of equal masses, situated in any manner in space, lines were drawn parallel and proportional to a system of the same number of forces in equilibrium, the centre of gravity of the system would remain unchanged by the transference of the several masses to the opposite extremities of the several lines.

This may be proved immediately as follows. The coordinates of the several masses being (x, y, z), (x', y', z'), &c., in their original, and (ξ, η, ζ), (ξ', η', ζ'), &c., in their transferred positions; since, from the equilibrium of the system of forces, $\Sigma(x-\xi) = 0$, $\Sigma(y-\eta) = 0$, $\Sigma(z-\zeta)$, therefore, m being their common mass, $\Sigma(mx) = \Sigma(m\xi)$, $\Sigma(my) = \Sigma(m\eta)$, $\Sigma(mz) = \Sigma(m\zeta)$, and therefore &c.

NOTE.—The above demonstration (for which, in the particular case in question, one purely geometrical might be easily substituted) establishes evidently the corresponding property in Geometry of Three Dimensions for a tetrahedron ABCD; the four points A′, B′, C′, D′ being the centres of gravity of the four faces, and the four lines $A'\alpha$, $B'\beta$, $C'\gamma$, $D'\delta$ being perpendicular and proportional to their four areas.

III. *Solution by* J. J. WALKER, M.A.

Let $A'\alpha = 2\lambda . B'C$, ..., then, if (x_1, y_1), ... (x_3, y_3) be the coordinates of A, B, C respectively, with respect to any two rectangular axes in the plane of ABC; (x', y'), ... (x''', y''') those of α, β, γ respectively; it easily appears from the figure that

$$x' = \tfrac{1}{2}(x_2 + x_3) + \tfrac{1}{2}\lambda(y_2 - y_3), \quad x'' = \tfrac{1}{2}(x_3 + x_1) + \tfrac{1}{2}\lambda(y_3 - y_1),$$
$$x''' = \tfrac{1}{2}(x_1 + x_2) + \tfrac{1}{2}\lambda(y_1 - y_2).$$

Consequently $\qquad \tfrac{1}{3}(x' + x'' + x''') = \tfrac{1}{3}(x_1 + x_2 + x_3).$

Similarly it may be proved that $\tfrac{1}{3}(y' + y'' + y''') = \tfrac{1}{3}(y_1 + y_2 + y_3)$; therefore, &c.

3047. (Proposed by the Rev. J. BLISSARD.)—To prove that

$$n - \frac{n(n-1)}{1.2}\left(1 - \frac{1}{2}\right) + \frac{n(n-1)(n-2)}{1.2.3}\left(1 - \frac{1}{2} + \frac{1}{3}\right) - \dots = \frac{2^n - 1}{n}.$$

I. *Solution by* J. A. MCNEILL.

$$n - \frac{n(n-1)}{1.2}\left(1 - \frac{1}{2}\right) + \frac{n(n-1)(n-2)}{1.2.3}\left(1 - \frac{1}{2} + \frac{1}{3}\right) - \dots\dots$$
$$= \left(n - \frac{n(n-1)}{1.2} + \frac{n(n-1)(n-2)}{1.2.3} - \dots\right) + \frac{1}{2}\left(\frac{n(n-1)}{1.2} - \frac{n(n-1)(n-2)}{1.2.3} + \dots\right)$$
$$+ \frac{1}{3}\left(\frac{n(n-1)(n-2)}{1.2.3} - \frac{n(n-1)(n-2)(n-3)}{1.2.3.4} + \dots\right) + \dots \quad \dots\dots(\alpha),$$

where the rth term is

$$=\frac{1}{r}\left\{\frac{n(n-1)\dots(n-r+1)}{1.2\dots r}-\frac{n(n-1)\dots(n-r)}{1.2\dots(r+1)}+\dots\dots\right\}$$

$$=(-1)^{r-1}.\frac{1}{r}\left\{1-n+\frac{n(n-1)}{1.2}-\frac{n(n-1)(n-2)}{1.2.3}\dots\dots\frac{n(n-1)\dots(n-r+2)}{1.2\dots r-1}\right\}$$

$$=\frac{1}{r}\left\{\text{coefficient of } x^{r-1} \text{ in } (1+x)^{n-1}\right\}=\frac{1}{r}\left\{\frac{(n-1)(n-2)\dots(n-r+1)}{1.2\dots r-1}\right\}$$

$$\therefore \quad (a)=1+\frac{1}{2}\left\{\frac{n-1}{1}\right\}+\dots\dots\frac{1}{r}\left\{\frac{(n-1)(n-2)\dots(n-r+1)}{1.2\dots(r-1)}\right\}+\dots\dots$$

$$=\frac{1}{n}\left\{n+\frac{n(n-1)}{1.2}+\frac{n(n-1)(n-2)}{1.2.3}+\dots\dots\right\}=\frac{2^n-1}{n}.$$

II. *Solution by the* PROPOSER.

The above Question, in a generalized form, may be solved as follows:— Using Representative Notation, let

$$U^n=\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\dots+\frac{x^n}{n} \quad \text{and} \quad R^n=\frac{1}{n};$$

then the theorem proved in the answer to Question 2928, viz.,

$$\frac{x}{1}+\frac{x^2}{2}+\dots+\frac{x^n}{n}=\frac{n}{1^2}\left\{1-(1-x)\right\}-\frac{n(n+1)}{1.2^2}\left\{1-(1-x)^2\right\}+\dots\dots,$$

becomes

$$U^n=\frac{nR}{1}\left\{1-(1-x)\right\}-\frac{n(n-1)}{1.2}R^2\left\{1-(1-x)^2\right\}+\dots\dots$$

$$=1-(1-R)^n-1+\left\{1-R(1-x)\right\}^n=\left\{1-R(1-x)\right\}^n-(1-R)^n.$$

Applying Taylor's Theorem, we have

$$f(h+U\theta)-fh=f\left\{h+\theta-R(1-x)\theta\right\}-f\left\{h+(1-R)\theta\right\}\dots\dots(1).$$

Let $h=1$, $\theta=-1$, and $fz=z^n$; then

$$(1-U)^n-1=R^n(1-x)^n-R^n=\frac{(1-x)^n-1}{n},$$

i.e., $-nU_1+\dfrac{n(n-1)}{1.2}U_2-\dots\dots$, which

$$=-n\frac{x}{1}+\frac{n(n-1)}{1.2}\left(\frac{x}{1}+\frac{x^2}{2}\right)-\frac{n(n-1)(n-2)}{1.2.3}\left(\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}\right)+\dots\dots$$

$$=\frac{(1-x)^n-1}{n}\dots\dots\dots\dots\dots(2).$$

Now let $x=-1$, and we have

$$n-\frac{n(n-1)}{1.2}\left(1-\frac{1}{2}\right)+\frac{n(n-1)(n-2)}{1.2.3}\left(1-\frac{1}{2}+\frac{1}{3}\right)+\dots\dots=\frac{2^n-1}{n}.$$

COR.—By equating coefficients of x^r in (2), we get

$$\frac{1}{n} = \frac{1}{r} - \frac{n-r}{r(r+1)} + \frac{(n-r)(n-r-1)}{r(r+1)(r+2)} - \dots\dots;$$

and putting $n+r$ for n,

$$\frac{1}{n+r} = \frac{1}{r} - \frac{n}{r(r+1)} + \frac{n(n-1)}{r(r+1)(r+2)} - \dots\dots.$$

NOTE.—Numerous results of a probably novel and interesting character may be obtained by varying the form of function in (1).

3104. (Proposed by R. TUCKER, M.A.)—Five numbers, a, b, c, d, e, are so related that a, b, c are in arithmetical progression; a, b, d in geometrical progression; and a, b, e in harmonical progression; also d increased by 10 is equal to the arithmetic mean between c and e, and the fourth proportional to c, d, e increased by 10 equals half their sum. Find the numbers.

I. *Solution by* H. HOSKINS; *the* PROPOSER; *and others.*

Let the five numbers be denoted by

$$a, \quad ar, \quad a(2r-1), \quad ar^2, \quad \text{and} \quad \frac{ar}{2-r}.$$

Therefore, by the question,

$$ar^2+10 = \tfrac{1}{2}a\left\{(2r-1)+\frac{r}{2-r}\right\} = a\,\frac{3r-r^2-1}{2-r} \dots\dots\dots\dots (1).$$

Also the fourth proportional to c, d, e is $\dfrac{ar^3}{(2r-1)(2-r)}$; whence

$$\frac{ar^3}{(2r-1)(2-r)}+10 = \tfrac{1}{2}a\left\{r^2+(2r-1)+\frac{r}{2-r}\right\} = \tfrac{1}{2}a\,\frac{6r-r^3-2}{(2-r)} \dots\dots (2).$$

Subtracting (2) from (1), we have $\dfrac{4(r^2-2r+1)}{(2r-1)(2-r)} = 1$; whence $r=\frac{3}{2}$ or $\frac{2}{3}$.

Hence the five numbers are,

if $r=\frac{3}{2}$, $\quad a, \ \frac{3}{2}a, \ 2a, \ \frac{9}{4}a, \ \text{and} \ 3a$;

if $r=\frac{2}{3}$, $\quad a, \ \frac{2}{3}a, \ \frac{1}{3}a, \ \frac{4}{9}a, \ \text{and} \ \frac{1}{2}a$.

Whence, from (1), we have the numerical values

40, 60, 80, 90, and 120; or -360, -240, -120, -160, and -180.

II. *Solution by* ARTEMAS MARTIN.

We have $\quad a+c=2b, \quad ad=b^2 \dots\dots\dots\dots\dots\dots\dots\dots (1, 2),$

$$\frac{2ae}{a+e}=b, \quad d+10=\tfrac{1}{2}(c+e), \quad \frac{de}{c}+10=\tfrac{1}{2}(c+d+e) \dots\dots (3, 4, 5).$$

Combining (4) and (5), we get $c = \frac{2}{3}e$. Substituting in (1) and (3),

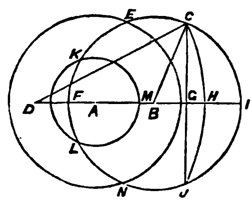
$$b = \tfrac{1}{2}(a + \tfrac{2}{3}e) = \tfrac{1}{2}a + \tfrac{1}{3}e \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

and $$\frac{2ae}{a+e} = \tfrac{1}{2}a + \tfrac{1}{3}e, \quad \text{or} \quad 2e^2 - 7ae = -3a^2;$$

whence $e = 3a$; therefore $b = \frac{3}{2}a$, $c = 2a$, $d = \frac{5}{2}a - 10$.
Substituting these values in (2), we find $a = 40$;
therefore $b = 60$, $c = 80$, $d = 90$, $e = 120$.

2700. (Proposed by ARTEMAS MARTIN.)—Three equal coins are piled at random on a horizontal plane; required the probability that the pile will stand.

Solution by the PROPOSER.



Let A be the centre of the first or bottom coin, and B the centre of the second coin. Take AD = AB, and with centre D and radius $2r$ (equal to twice the radius of one of the equal coins) describe the arc CHJ. With centre A and radius $\frac{1}{2}r$ describe the circle KLM. If the centre of the second coin is within this circle, the pile will stand, if the centre of the third or top coin is anywhere on the second, since in that case the common centre of gravity of the second and third coins falls on the bottom one.

The probability that the centre of the second coin will fall within the circle KLM is $\dfrac{\frac{1}{4}\pi r^2}{4\pi r^2} = \dfrac{1}{16}$, since its centre may be anywhere in the circle whose centre is A and radius $2r$; and the probability that the centre of the third coin will fall on the second is $\dfrac{\pi r^2}{4\pi r^2} = \dfrac{1}{4}$.

Hence the probability that the centre of the second coin will fall within the circle KLM, and the centre of the third coin fall on the second, is $\dfrac{1}{16} \times \dfrac{1}{4} = \dfrac{1}{64}$.

If the centre of the second coin be without the circle KLM, as at B, the pile will stand if the centre of the third or top coin is anywhere on the surface CHJNLFKE (= S, suppose), as in that case the common centre of gravity of the second and third coins falls on the bottom one.

The probability that the centre of the second coin will be on the first and without the circle KLM, is $\dfrac{1}{4} - \dfrac{1}{16} = \dfrac{3}{16}$; the probability that the centre of the third coin will fall on the surface S is $\dfrac{S}{4\pi r^2}$; and the proba-

bility that the centre of the second coin will fall on the first and without the circle KLM, and the centre of the third or top coin fall on S, is

$$\frac{3}{16}\cdot\frac{S}{4\pi r^2}=\frac{3S}{64\pi r^2}.$$

Let θ be the angle CDB, ϕ the angle CBG, ψ the angle DCB, and $x=$AB. Then DB $=2x$, and we have

$$4x^2+4r^2-8rx\cos\theta=r^2\ldots\ldots(1),\quad 4x^2+r^2+4rx\cos\phi=4r^2\ldots\ldots(2),$$

and
$$4r^2+r^2-4r^2\cos\psi=4x^2\ldots\ldots(3).$$

These equations give

$$\cos\theta=\frac{3r^2+4x^2}{8rx},\quad \cos\phi=\frac{3r^2-4x^2}{4rx},\quad \cos\psi=\frac{5r^2-4x^2}{4r^2}\ldots\ldots(4).$$

Now we have
$$\text{Segment CHJG}=4r^2(\theta-\sin\theta\cos\theta),$$
$$\text{Segment CIJG}=r^2(\phi-\sin\phi\cos\phi),$$
$$\text{Lune CIJH}=r^2(\phi-4\theta-\sin\phi\cos\phi+4\sin\theta\cos\theta),$$
$$\text{CHJNLFKE}=\pi r^2-r^2(\phi-4\theta-\sin\phi\cos\phi+4\sin\theta\cos\theta)=S.$$

Hence, if p be the probability required, we have

$$p=\frac{1}{64}+\frac{\int_r^{2r} 3S\,.\,2\pi x\,dx}{64\pi r^2\int_r^{2r}2\pi x\,dx}=\frac{1}{64}+\frac{1}{8\pi r^4}\int_r^{2r}Sx\,dx$$

$$=\frac{1}{16}-\frac{1}{8\pi r^2}\int_r^{2r}(\phi-4\theta-\sin\phi\cos\phi+4\sin\theta\cos\theta)\,x\,dx.$$

By the help of the relations in (4), we readily find

$$\int\phi x\,dx=\tfrac{1}{2}x^2\phi-\tfrac{1}{2}r^2\psi+\tfrac{1}{2}r^2\sin\psi,$$
$$\int 4\theta x\,dx=2x^2\theta+\tfrac{1}{2}r^2\psi-r^2\sin\psi,$$
$$\int(4\sin\theta\cos\theta-\sin\phi\cos\phi)\,x\,dx=\int\sin\phi\,(2\cos\theta-\cos\phi)\,x\,dx,$$

since $\sin\phi=2\sin\theta$, $=\tfrac{1}{2}r^2\psi-\tfrac{1}{2}r^2\sin\psi\cos\psi$; therefore

$$p=\frac{1}{16}-\frac{1}{8\pi r^2}\Big[\tfrac{1}{2}x^2\phi-2x^2\theta-\tfrac{1}{2}r^2\psi+\tfrac{3}{2}r^2\sin\psi-\tfrac{1}{2}r^2\sin\psi\cos\psi\Big]_r^{2r}$$
$$=\frac{1}{16}-\frac{3}{16\pi}\left(\frac{3}{16}\sqrt{15}-2\sin^{-1}\frac{1}{4}\right).$$

This agrees with the result obtained by Mr Woolhouse on p. 35 of this volume, if we make an equation [illegible] of that solution the obvious correction of 3 for 8.

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bility that the centre of the second coin will fall on the first and without the circle KLM, and the centre of the third or top coin fall on S, is

$$\frac{3}{16} \cdot \frac{S}{4\pi r^2} = \frac{3S}{64\pi r^2}.$$

Let θ be the angle CDB, ϕ the angle CBG, ψ the angle DCB, and $x = AB$. Then $DB = 2x$, and we have

$$4x^2 + 4r^2 - 8rx\cos\theta = r^2 \ldots\ldots (1), \quad 4x^2 + r^2 + 4rx\cos\phi = 4r^2 \ldots\ldots (2),$$

and

$$4r^2 + r^2 - 4r^2\cos\psi = 4x^2 \ldots\ldots (3).$$

These equations give

$$\cos\theta = \frac{3r^2 + 4x^2}{8rx}, \quad \cos\phi = \frac{3r^2 - 4x^2}{4rx}, \quad \cos\psi = \frac{5r^2 - 4x^2}{4r^2} \ldots\ldots (4).$$

Now we have

$$\text{Segment CHJG} = 4r^2(\theta - \sin\theta\cos\theta),$$

$$\text{Segment CIJG} = r^2(\phi - \sin\phi\cos\phi),$$

$$\text{Lune CIJH} = r^2(\phi - 4\theta - \sin\phi\cos\phi + 4\sin\theta\cos\theta),$$

$$\text{CHJNLFKE} = \pi r^2 - r^2(\phi - 4\theta - \sin\phi\cos\phi + 4\sin\theta\cos\theta) = S.$$

Hence, if p be the probability required, we have

$$p = \frac{1}{64} + \frac{\int_{\frac{1}{2}r}^{r} 3S \, . \, 2\pi x\,dx}{64\pi r^2 \int_{\frac{1}{2}r}^{r} 2\pi x\,dx} = \frac{1}{64} + \frac{1}{8\pi r^4}\int_{\frac{1}{2}r}^{r} Sx\,dx$$

$$= \frac{1}{16} - \frac{1}{8\pi r^2}\int_{\frac{1}{2}r}^{r} (\phi - 4\theta - \sin\phi\cos\phi + 4\sin\theta\cos\theta)\,x\,dx.$$

By the help of the relations in (4), we readily find

$$\int \phi x\,dx = \tfrac{1}{2}x^2\phi - \tfrac{1}{2}r^2\psi + \tfrac{1}{4}r^2\sin\psi,$$

$$\int 4\theta x\,dx = 2x^2\theta + \tfrac{1}{2}r^2\psi - r^2\sin\psi,$$

$$\int (4\sin\theta\cos\theta - \sin\phi\cos\phi)\,x\,dx = \int \sin\phi\,(2\cos\theta - \cos\phi)\,x\,dx,$$

since $\sin\phi = 2\sin\theta = \frac{1}{2}r^2\psi - \frac{1}{2}r^2\sin\psi\cos\psi$; therefore

$$p = \frac{1}{16} - \frac{1}{8\pi r^2}\left[\tfrac{1}{2}x^2\phi - 2x^2\theta - \tfrac{1}{2}r^2\psi + \tfrac{5}{4}r^2\sin\psi - \tfrac{1}{4}r^2\sin\psi\cos\psi\right]_{\frac{1}{2}r}^{r}$$

$$= \frac{1}{16} - \frac{3}{16\pi}\left(\frac{3}{16}\sqrt{15} - 2\sin^{-1}\frac{1}{4}\right).$$

[This agrees with the result obtained by Mr. Woolhouse on p. 36 of this volume, if we make in equation (γ) of that solution the obvious correction of 3 for 9.]

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